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Sparsity-promoting optimal control of consensus and synchronization networks

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Abstract— **We study a class of optimal control problems that are encountered in consensus and synchronization networks. These are characterized by structural constraints that arise from the lack of the absolute measurements for a part of the state vector. In order to deal with these constraints, we introduce a coordinate transformation to eliminate the average mode and assure stabilizability and detectability of the remaining modes. To design networks with low communication requirements, we seek solutions to the ℓ_1 regularized version of the standard $\mathcal{H}_2$ optimal control problem. Such solutions trade off network performance and sparsity of the controller. We also identify a class of optimal control problems that can be cast as semidefinite programs. Examples from power systems and consensus networks are provided to illustrate our developments.**

Index Terms— **Alternating direction method of multipliers, consensus, power systems, semidefinite programming, sparsity-promoting optimal control, synchronization, wide-area control.**

I. Introduction

Consensus and synchronization problems are of increased importance in networks of dynamical systems [1]. These problems are encountered in a number of applications ranging from biology to computer science to power systems [2]–[8]. In each of these applications, it is of interest to reach an agreement or to achieve synchronization by exchanging relative information between the nodes. The restriction on the absence of the absolute measurements imposes structural constraints for the analysis and design.

Conventional optimal control of distributed systems relies on centralized implementation of control policies [9]. In large networks of dynamical systems centralized information processing may impose heavy communication and computation burden on individual nodes. This motivates the development of localized feedback control strategies that require limited information exchange between the nodes in order to reach consensus or guarantee synchronization [10]–[15].

The objective of this paper is to design optimal controllers that provide a desirable tradeoff between the network performance and sparsity of the controller. A coordinate transformation is introduced to eliminate the unobservable average mode and bring the network dynamics in a form that is amenable to both standard and sparsity-promoting optimal control methods [16], [17]. This facilitates identification of sparse control architectures that are capable of achieving the desired consensus or synchronization performance. In the context of wide-area control of power systems [18], [19], the optimal controller respects the structure of the original power network: in both open- and closed-loop systems, only relative rotor angle differences between different generators appear in the state-space representation.

Our paper is organized as follows. In Section II, we describe a class of optimal control problems that we examine. In Section III, we formulate the sparsity-promoting consensus and synchronization optimal control problems and solve them using the alternating direction method of multipliers; we also identify a class of design problems that can be formulated as semidefinite programs. In Section IV, we illustrate our developments using examples of consensus and power networks. Finally, we provide a brief summary of our results in Section V.

II. Motivation and background

A. Problem setup

We consider a class of control problems

$$\begin{aligned}
\begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} &= \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} + \\
& \quad \begin{bmatrix} \bar{B}_{1p} \\ \bar{B}_{1q} \end{bmatrix} d + \begin{bmatrix} \bar{B}_{2p} \\ \bar{B}_{2q} \end{bmatrix} u \\
z &= \begin{bmatrix} Q_p^{1/2} & 0 \\ 0 & Q_q^{1/2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ R^{1/2} \end{bmatrix} u \\
u &= -\begin{bmatrix} G_p & G_q \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}
\end{aligned} \tag{1}$$

where d and u denote the disturbance and control inputs, and z is the performance output. The feedback gains G_p and G_q are the design variables under the standard assumptions on the state and control weights $Q_p = Q_p^T \succeq 0$, $Q_q = Q_q^T \succeq 0$, and $R = R^T \succ 0$.

We restrict our attention to the problems in which only relative differences between the components of the vector $p(t) \in \mathbb{R}^N$ enter into (1). This requirement imposes structural constraints on the matrices in (1), which are partitioned conformably with the partition of the state vector

$$\phi := \begin{bmatrix} p \\ q \end{bmatrix} \in \mathbb{R}^n.$$

This restriction on the absence of the access to the absolute measurements of the components of the vector p translates

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into the following requirements

$$\begin{array}{rclrcl} \bar{A}_{11}\,\mathbb{1} & = & 0, & \bar{A}_{21}\,\mathbb{1} & = & 0 \\ Q_p\,\mathbb{1} & = & 0, & G_p\,\mathbb{1} & = & 0 \end{array} \tag{2}$$

where $\mathbb{1}$ is the vector of all ones. Furthermore, since our primary control objective is to achieve agreement between the components of the vector $p(t) \in \mathbb{R}^N$ we will assume that the matrix Q_p quantifies the deviation from average,

$$Q_p \;=\; I \,-\, (1/N)\,\mathbb{1}\mathbb{1}^T. \tag{3}$$

If the submatrices in (1) satisfy (2), then both the open-loop system (1) and the closed-loop system,

$$\begin{array}{rcl} \dot{\phi} & = & (\bar{A} \,-\, \bar{B}_2\,G)\,\phi \,+\, \bar{B}_1\,d \\ z & = & \begin{bmatrix} \bar{Q}^{1/2} \\ -R^{1/2}\,G \end{bmatrix}\phi \end{array} \tag{4}$$

have an eigenvalue at zero

$$\begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}\begin{bmatrix} \mathbb{1} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \bar{A}_{11} - \bar{B}_{2p}G_p & \bar{A}_{12} - \bar{B}_{2p}G_q \\ \bar{A}_{21} - \bar{B}_{2q}G_p & \bar{A}_{22} - \bar{B}_{2q}G_q \end{bmatrix}\begin{bmatrix} \mathbb{1} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{5}$$

and the corresponding eigenvector is associated with the average of the vector p,

$$\bar{p} \;:=\; (1/N)\,\mathbb{1}^T p.$$

We assume stabilizability and detectability of the pairs $(\bar{A}, \bar{B}_2)$ and $(\bar{A}, \bar{Q}^{1/2})$ on the subspace $\mathcal{S}$,

$$\mathcal{S} \;:=\; \begin{bmatrix} \mathbb{1} \\ 0 \end{bmatrix}^{\perp} = \begin{bmatrix} \mathbb{1}^{\perp} \\ \mathbb{R}^{n-N} \end{bmatrix}$$

and formulate the optimal control problem in Section III.

B. Examples

To motivate our developments, we next provide several examples that can be represented in the form described in Section II-A.

1) Swing equation: In power networks, swing equation is used to characterize energy exchange between generators [20]. After linearization at a stationary operating point, the swing equation reduces to

$$M\,\ddot{p} \,+\, D\,\dot{p} \,+\, L_p\,p \,=\, d \,+\, u \tag{6}$$

where M and D are diagonal matrices of inertia and damping coefficients, and L_p is the Laplacian matrix that describes interactions between different nodes. By setting $q := \dot{p}$, the swing equation (6) is brought into the state-space form (1) with

$$\bar{A} \,=\, \begin{bmatrix} 0 & I \\ -M^{-1}\,L_p & -M^{-1}\,D \end{bmatrix}, \; \bar{B}_i \,=\, \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix}.$$

Furthermore, networks in which each node updates a scalar variable p_i using relative information exchange with its neighbors can be obtained from (6) by setting the matrix M to zero. In this case, the state-space model (1) simplifies to

$$\bar{A} \;=\; -D^{-1}\,L_p, \;\; \bar{B}_1 \;=\; \bar{B}_2 \;=\; D^{-1}. \tag{7}$$

Since, by definition $L_p\mathbb{1} = 0$, the structural restrictions (2) are satisfied if $Q_p\mathbb{1} = 0$ and $G_p\mathbb{1} = 0$.

2) Power systems: Models of power networks account for the dynamics of generators, control devices, and algebraic equations that describe load flow, stators, and electronic circuits. Control actions are typically executed using generator excitation via power system stabilizers (PSS), governor control, or power electronics (FACTS) [21]. Thus, in addition to the rotor angles p and frequencies $v := \dot{p}$, additional states r that account for fast electrical dynamics are needed to describe the dynamics of the entire system. After linearization at a stationary operating point, the state-space model can be written in the form (1) by defining

$$q \;:=\; \begin{bmatrix} v \\ r \end{bmatrix}$$

with

$$\begin{array}{rclrcl} \bar{A}_{11} & = & 0, & \bar{A}_{21} & = & \begin{bmatrix} -M^{-1}\,L_p \\ \bar{A}_{rp} \end{bmatrix} \\ \bar{A}_{12} & = & \begin{bmatrix} I & 0 \end{bmatrix}, & \bar{A}_{22} & = & \begin{bmatrix} -M^{-1}\,D & \bar{A}_{qr} \\ \bar{A}_{rq} & \bar{A}_{rr} \end{bmatrix}. \end{array}$$

Since only differences between rotor angles of different generators enter into the original nonlinear differential equations, this property is shared by the linearized set of equations, thereby implying $\bar{A}_{21}\mathbb{1} = 0$. Furthermore, in the absence of the access to the absolute rotor angle measurements both the matrix $\bar{A}$ in (1) and its closed-loop equivalent satisfy properties specified in (5).

III. Sparsity-promoting optimal control

A. Elimination of the average mode

To eliminate the average-mode, $\bar{p} := (1/N)\,\mathbb{1}^T p$, from (4) we introduce the following coordinate transformation

$$\underbrace{\begin{bmatrix} p \\ q \end{bmatrix}}_{\phi} = \underbrace{\begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix}}_{T}\underbrace{\begin{bmatrix} \psi \\ q \end{bmatrix}}_{x} + \begin{bmatrix} \mathbb{1} \\ 0 \end{bmatrix}\bar{p}$$

where the columns of the matrix $U \in \mathbb{R}^{N\times(N-1)}$ form an orthonormal basis for the subspace $\mathbb{1}^{\perp}$. In particular, the columns of U are obtained from the $(N-1)$ eigenvectors of the matrix Q_p in (3) corresponding to the non-zero eigenvalues. Using properties of the matrix U

$$U^T\,U \;=\; I, \;\; U\,U^T \;=\; I \,-\, (1/N)\,\mathbb{1}\mathbb{1}^T, \;\; U^T\,\mathbb{1} \;=\; 0$$

we equivalently have

$$x \;:=\; \begin{bmatrix} \psi \\ q \end{bmatrix} = \begin{bmatrix} U^T & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} p \\ q \end{bmatrix} \;=\; T^T\phi.$$

State-space representation of the closed-loop system (4) on the subspace $\mathcal{S}$ is obtained by combining the above relations between ϕ and x with (5),

$$
\begin{aligned}
\dot{x} &= (A - B_2 F)\, x \,+\, B_1\, d \\
z &= \begin{bmatrix} Q^{1/2} \\ -R^{1/2}\, F \end{bmatrix} x
\end{aligned}
\qquad (8)
$$

where

$$A \;:=\; T^T \bar{A}\, T, \;\; B_i \;:=\; T^T\, \bar{B}_i, \;\; Q^{1/2} \;:=\; \bar{Q}^{1/2}\, T.$$

Furthermore, using the properties of the matrix U it is readily shown that the feedback gain matrices F and G are related by

$$F \;=\; G\,T \quad \Leftrightarrow \quad G \;=\; F\,T^T.$$

Our control objective is to achieve a desirable tradeoff between the $\mathcal{H}_2$ performance of (8) and the sparsity of the feedback gain G. The $\mathcal{H}_2$ norm from d to z is defined as

$$J(F) \;:=\; \begin{cases} \operatorname{trace}\left(B_1^T\, P(F)\, B_1\right) & F \text{ stabilizing} \\ \infty & \text{otherwise} \end{cases}$$

where the closed-loop observability Gramian $P(F)$ satisfies the Lyapunov equation

$$(A \,-\, B_2\, F)^T P \,+\, P\, (A \,-\, B_2\, F) \;=\; -(Q \,+\, F^T R\, F).$$

While quadratic performance is expressed in terms of the feedback gain matrix F, as we describe in Section III-B, it is desired to enhance sparsity of the feedback gain matrix G in the original set of coordinates.

B. Sparsity-promoting optimal control

We now identify feedback gains that achieve a desirable tradeoff between variance amplification of the closed-loop system (8) and sparsity of the controller. After the desired tradeoff has been obtained, we fix the sparsity structure and determine the optimal feedback gain that minimizes the $\mathcal{H}_2$ norm of (8). Our approach augments the developments of [17] and it utilizes the alternating direction method of multipliers (ADMM) algorithm.

While variance minimization is most conveniently expressed in terms of the feedback gain F, it is desirable to promote sparsity of the feedback gain G in the original set of coordinates. To address this challenge we consider a regularized $\mathcal{H}_2$ optimal control problem

$$
\begin{array}{ll}
\text{minimize} & J(F) \,+\, \gamma\, g(G) \\
\text{subject to} & F\,T^T \,-\, G \;=\; 0
\end{array}
\qquad (9)
$$

where the regularization term is determined by the weighted ℓ_1 norm,

$$g(G) \;:=\; \sum_{i,\,j} W_{ij}\, |G_{ij}|.$$

We note that the linear constraint in (9) is more general than the constraint considered in [17].

In order to provide better approximation of the cardinality function $\textbf{card}\,(G)$, which determines the number of the non-zero elements in G, the weights W_{ij} are selected using the procedure introduced in [22]. The positive regularization parameter γ specifies the importance of sparsity. For $\gamma = 0$, the standard centralized linear quadratic regulator is obtained; as γ increases, the feedback matrix G becomes sparser.

As we describe next, the introduction of the linear constraint in (9) in conjunction with the ADMM algorithm allows us to exploit structure of the objective functions J and g in (9).

The ADMM algorithm is described next; we refer the reader to [17] for additional details.

1) Form augmented Lagrangian:

$$
\begin{aligned}
\mathcal{L}_\rho(F, G, \Lambda) \;=\; & J(F) \,+\, \gamma\, g(G) \,+ \\
& \operatorname{trace}\left(\Lambda^T (F\,T^T \,-\, G)\right) \,+\, \frac{\rho}{2}\, \|\, F\,T^T - G\,\|_F^2
\end{aligned}
$$

where Λ denotes the matrix of Lagrange multipliers and $\|\cdot\|_F$ is the Frobenius norm of a matrix.

2) Iterative ADMM algorithm:

$$
\begin{aligned}
F^{k+1} &= \operatorname*{arg\,min}_F \, \mathcal{L}_\rho\, (F,\, G^k,\, \Lambda^k) \\
G^{k+1} &= \operatorname*{arg\,min}_G \, \mathcal{L}_\rho\, (F^{k+1},\, G,\, \Lambda^k) \\
\Lambda^{k+1} &= \Lambda^k \,+\, \rho\, (F^{k+1}\, T^T \,-\, G^{k+1}).
\end{aligned}
$$

Using the fact that $T^T T \;=\; I$, it is readily shown that the F-minimization step amounts to solving the following optimization problem

$$F^{k+1} \;=\; \operatorname*{arg\,min}_F \left(J(F) \,+\, \frac{\rho}{2}\, \|F \,-\, H^k\|_F^2 \right)$$

where

$$H^k \;:=\; (G^k \,-\, (1/\rho)\, \Lambda^k)\, T.$$

As shown in [17], the necessary conditions for the optimality of $\mathcal{L}_\rho(F, G^k, \Lambda^k)$ are given by

$$
\begin{aligned}
(A - B_2 F) L \,+\, L (A - B_2 F)^T &= -B_1 B_1^T \\
(A - B_2 F)^T P \,+\, P (A - B_2 F) &= -(Q + F^T R F), \\
2(RF - B_2^T P) L \,+\, \rho (F \,-\, H^k) &= 0.
\end{aligned}
$$

The resulting set of the matrix-valued equations is solved using the iterative procedure developed in [17].

Similarly, properties of the matrix T can be used to bring the G-minimization problem into the following form

$$G^{k+1} \;=\; \operatorname*{arg\,min}_G \left(\gamma\, g(G) \,+\, \frac{\rho}{2}\, \|G \,-\, V^k\|_F^2 \right)$$

where

$$V^k \;:=\; F^{k+1}\, T^T + (1/\rho)\, \Lambda^k$$

and the unique solution is obtained via convenient use of the soft thresholding operator,

$$G_{ij}^{k+1} \;=\; \begin{cases} (1 \,-\, a/|V_{ij}^k|)\, V_{ij}^k & |V_{ij}^k| \,>\, a \\ 0 & |V_{ij}^k| \,\leq\, a. \end{cases}$$

Here, $a \;:=\; (\gamma/\rho) W_{ij}$ and, for a given V_{ij}^k, G_{ij}^{k+1} is either set to zero or it is obtained by moving V_{ij}^k towards zero with the amount $(\gamma/\rho) W_{ij}$.

3) Stopping criterion:

$$\begin{aligned} \|F^{k+1} T^T - G^{k+1}\| &\leq \epsilon \\ \|G^{k+1} - G^k\| &\leq \epsilon \end{aligned}$$

The ADMM algorithm stops when both primal and dual residuals are smaller than specified thresholds.

4) Polishing step: Finally, we fix the sparsity pattern of G identified using ADMM and solve the optimal control problem in the presence of the identified structural constraints. This polishing step improves the $\mathcal{H}_2$ performance relative to the feedback gain identified by ADMM; see [17] for additional details.

C. Class of convex problems

For an undirected network in which each node updates a scalar value p_i, we next show that the sparsity promoting optimal control problem can be formulated as an SDP. The closed-loop system (4) with

$$\bar{A} := -L_p, \quad \bar{B}_1 = \bar{B}_2 := I, \quad G := L_k$$

can be written as

$$\begin{aligned} \dot{p} &= -(L_p + L_k)\, p + d \\ z &= \begin{bmatrix} Q^{1/2} \\ -R^{1/2} L_k \end{bmatrix} p \end{aligned} \tag{10}$$

where L_p and L_k satisfy

$$\begin{aligned} L_p \mathbb{1} &= 0, \quad L_p = L_p^T \succeq 0 \\ L_k \mathbb{1} &= 0, \quad L_k = L_k^T \succeq 0. \end{aligned}$$

These two Laplacian matrices contain information about the interconnection structure of the open-loop system and the feedback gain, respectively.

The ℓ_1-regularized $\mathcal{H}_2$ optimal control problem can be formulated as

$$\underset{L_k}{\text{minimize}} \quad J(L_k) + \gamma \|W \circ L_k\|_{\ell_1} \tag{11}$$

where $\circ$ denotes elementwise multiplication of two matrices. On the other hand, the solution to the algebraic Lyapunov equation

$$(L_p + L_k)\, P + P\, (L_p + L_k) = Q + L_k R L_k$$

can be used to compute the variance amplification of the closed-loop system, $J(L_k) = \text{trace}\,(P)$. It is readily shown that the stability of (10) on the subspace $\mathbb{1}^\perp$ amounts to positive-definiteness of the matrix $(L_p + L_k)$ on $\mathbb{1}^\perp$. Under this condition, we can rewrite $J(L_k)$ as

$$\frac{1}{2} \text{trace}\, \left((L_p + L_k + (1/N)\, \mathbb{1}\mathbb{1}^T)^{-1} (Q + L_k R L_k) \right)$$

and cast the sparsity-promoting optimal control problem (11) as an SDP,

$$\begin{aligned} \underset{X,Y,Z,L_k}{\text{minimize}} \quad & \frac{1}{2} \text{trace}\,(X + Y) + \gamma \mathbb{1}^T Z \mathbb{1} \\ \text{subject to} \quad & \begin{bmatrix} X & Q^{1/2} \\ Q^{1/2} & L_p + L_k + (1/N)\, \mathbb{1}\mathbb{1}^T \end{bmatrix} \succeq 0 \\ & \begin{bmatrix} Y & R^{1/2} L_k \\ L_k R^{1/2} & L_p + L_k + (1/N)\, \mathbb{1}\mathbb{1}^T \end{bmatrix} \succeq 0 \\ & L_k \mathbb{1} = 0 \\ & -Z \leq W \circ L_k \leq Z. \end{aligned} \tag{12}$$

For small and medium size problems, the resulting SDP formulation can be solved efficiently using available SDP solvers.

IV. Examples

A. IEEE 14 bus test case

For IEEE 14 bus benchmark problem, sparsity-promoting optimal control problem is formulated for the system (8), in the set of coordinates where the average-mode is eliminated. This removes a requirement on introducing small regularization penalty on absolute rotor angles that was recently made in [18], [19]. As shown in Fig 1, 14 buses connect 2 generators and 3 synchronous condensers. A synchronous condenser is a device identical to a synchronous generator whose shaft is not connected to anything but spins freely. Its purpose is not to convert electric power to mechanical power or vice versa, but to adjust conditions on the electric power transmission grid. A voltage regulator is used to either generate or absorb reactive power as needed to adjust the grid's voltage, or to improve power factor. In this test case, we treat synchronous condensers as generators.

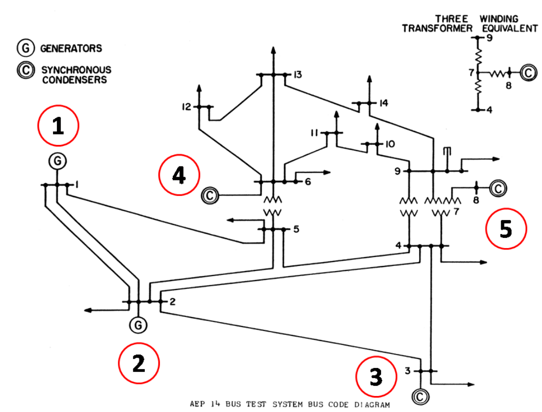
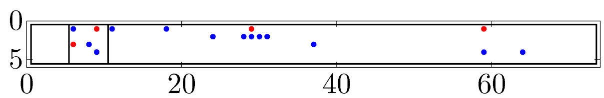


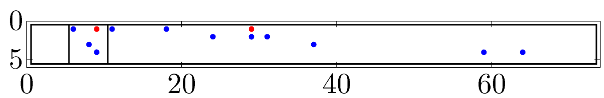
Fig. 1: IEEE 14 bus test case.

We apply control signal to the exciter of each generator to control the rotor angle and frequency. Exciter system is designed to control the applied voltage, and thus the field current to the rotor, which in turn gives control of generator output or terminal voltage. Subsequently this is what provides reactive power and power factor control between the generator and the system. Conventionally, control inputs are applied on excitation system, because they are easier to operate and are characterized by small time constants.

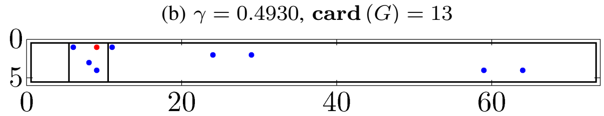
In our problem formulation, local exciter controllers are already embedded into the open-loop system. Our objective is to find a trade-off between $\mathcal{H}_2$ performance and sparsity of the feedback gain. In Fig. 2, we illustrate the sparsity pattern of the feedback matrix G in the original set of coordinates for different values of γ. Five rows correspond to five control inputs of each generator. The first block corresponds to the rotor angles p, the second block corresponds to rotor frequencies $v := \dot{p}$, and the third block corresponds to the remaining states r. The blue dots represent fully-decentralized feedback gains, and the red dots represent information exchange between different generators.



(a) $\gamma = 0.2705$, $\mathbf{card}\,(G) = 17$



(b) $\gamma = 0.4930$, $\mathbf{card}\,(G) = 13$



(c) $\gamma = 2$, $\mathbf{card}\,(G) = 9$

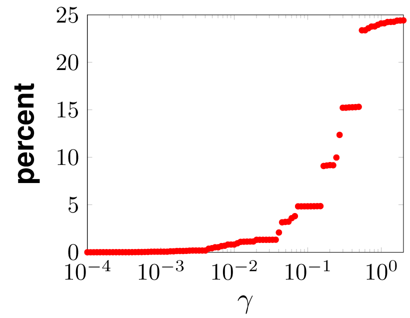
Fig. 2: Sparsity pattern of G for the exciter control.

When $\gamma = 0$, the optimal centralized feedback gain G_c is a dense matrix populated with nonzero elements. When $\gamma = 0.4930$, controller on generator 1 needs to have access to frequency of generator 4 and also the exciter information from generator 2; see Fig. 2a. Relative to the optimal centralized controller, sparse controller with only two long-range communication links compromises performance by only about 15%. The nonzero elements corresponding to the local states indicate feedback gains that are retuned by our procedure. When we increase γ to 2, only one long-range communication link remains and the resulting performance drop is 24.4%.

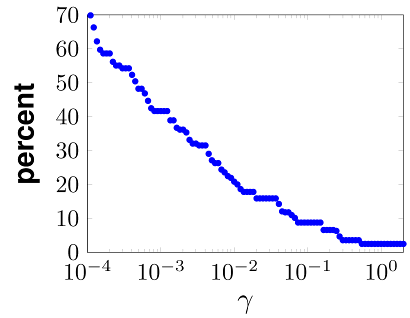
In Fig. 3, we can see that, relative to the optimal centralized controller G_c, performance of the optimal sparse controller decreases with increased emphasis on sparsity. As expected, increased emphasis on sparsity induces feedback gains with smaller number of nonzero elements.

B. Single-integrator network

In Fig. 4, 30 nodes are randomly distributed in a unit square. The graph Laplacian of the open-loop system L_p is obtained based on proximity of two nodes: a pair of nodes communicate with each other if their distance is not greater than 0.25 units. Each node is equipped with a controller that is designed to minimize the total mean-square deviation from average of the closed-loop network.



(a) $(J - J_c)/J_c$



(b) $\mathbf{card}\,(G)/\mathbf{card}\,(G_c)$

Fig. 3: Performance drop and the number of non-zero feedback gains for the exciter control.

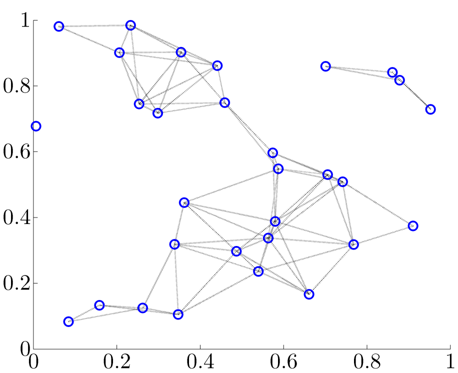


Fig. 4: Local performance graph, where blue edges connect pair of nodes with distance less than 0.25

Using the SDP characterization (12) of the sparsity-promoting optimal control problem (11), the optimal L_k is computed in CVX [23]. Figure 5 illustrates that when $\gamma = 0.0621$, only 7 additional long-range communication links are needed to achieve performance which is within 26% of the optimal centralized gain. Furthermore, for the smallest value of γ in Fig. 6, we see that dropping 25% of links from the centralized controller almost yields optimal centralized performance. On the other hand, for the largest value of γ in Fig. 6, we observe that the sparse controller with only 1.6% of links present in the centralized controller compromises performance by about 46%.

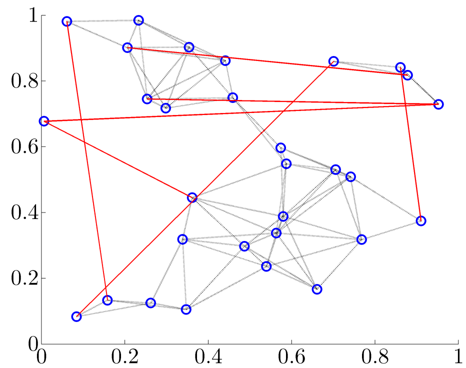
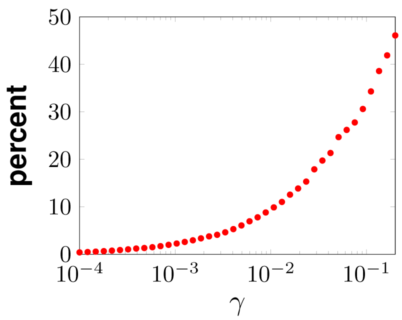
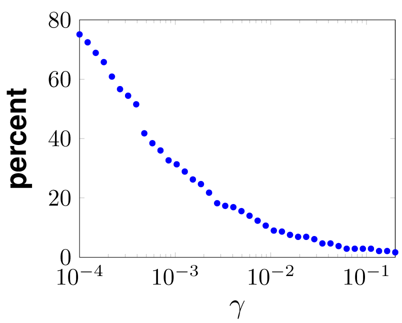


Fig. 5: Identified communication graph, where long-range communication links are highlighted in red.



(a) $(J - J_c)/J_c$



(b) $\mathbf{card}(G)/\mathbf{card}(G_c)$

Fig. 6: Performance drop and the number of non-zero feedback gains.

V. Concluding Remarks

In this paper, we consider ℓ_1 regularized version of the standard $\mathcal{H}_2$ optimal control problem for consensus and synchronization networks. We achieve desirable performance with fewer controller links by promoting sparsity on the feedback matrix. Alternating direction method of multipliers allows for performance and sparsity requirements to be expressed in different set of coordinates which allows efficient computation of the optimal controllers. The developments are illustrated using examples from power and consensus networks.

Acknowledgments

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References

[1] M. Mesbahi and M. Egerstedt, *Graph Theoretic Methods in Multiagent Networks*. Princeton University Press, 2010.

[2] F. Dörfler and F. Bullo, "Synchronization and transient stability in power networks and non-uniform Kuramoto oscillators," *SIAM Journal on Control and Optimization*, vol. 50, pp. 1616–1642, 2012.

[3] F. Dörfler and F. Bullo, "Exploring synchronization in complex oscillator networks," in *Proceedings of the 51st IEEE Conference on Decision and Control*, 2012, pp. 7157–7170.

[4] A. Mauroy, P. Sacré, and R. J. Sepulchre, "Kick synchronization versus diffusive synchronization," in *Proceedings of the 51st IEEE Conference on Decision and Control*, 2012, pp. 7171–7183.

[5] M. Arcak, "Synchronization and pattern formation in diffusively coupled systems," in *Proceedings of the 51st IEEE Conference on Decision and Control*, 2012, pp. 7184–7192.

[6] F. Dörfler, M. Chertkov, and F. Bullo, "Synchronization assessment in power networks and coupled oscillators," in *Proceedings of the 51st IEEE Conference on Decision and Control*, 2012, pp. 4998–5003.

[7] F. Dörfler and F. Bullo, "On the critical coupling strength for Kuramoto oscillators," in *Proceedings of the 2011 American Control Conference*, 2011, pp. 3239–3244.

[8] W. Ren, "Synchronization of coupled harmonic oscillators with local interaction," *Automatica*, vol. 44, pp. 3195–3200, 2008.

[9] G. E. Dullerud and F. Paganini, *A course in robust control theory*. Springer, 2000.

[10] L. Xiao, S. Boyd, and S.-J. Kim, "Distributed average consensus with least-mean-square deviation," *J. Parallel Distrib. Comput.*, vol. 67, no. 1, pp. 33–46, 2007.

[11] A. Ghosh, S. Boyd, and A. Saberi, "Minimizing effective resistance of a graph," *SIAM Review*, vol. 50, no. 1, pp. 37–66, 2008.

[12] D. Zelazo and M. Mesbahi, "Edge agreement: Graph-theoretic performance bounds and passivity analysis," *IEEE Trans. Automat. Control*, vol. 56, no. 3, pp. 544–555, 2011.

[13] B. Bamieh, M. R. Jovanović, P. Mitra, and S. Patterson, "Coherence in large-scale networks: dimension dependent limitations of local feedback," *IEEE Trans. Automat. Control*, vol. 57, no. 9, pp. 2235–2249, September 2012.

[14] D. Zelazo, S. Schuler, and F. Allgöwer, "Performance and design of cycles in consensus networks," *Syst. Control Lett.*, vol. 62, no. 1, pp. 85–96, 2013.

[15] M. Fardad, F. Lin, and M. R. Jovanović, "Design of optimal sparse interconnection graphs for synchronization of oscillator networks," *IEEE Trans. Automat. Control*, 2014, doi:10.1109/TAC.2014.2301577.

[16] M. Fardad, F. Lin, and M. R. Jovanović, "Sparsity-promoting optimal control for a class of distributed systems," in *Proceedings of the 2011 American Control Conference*, San Francisco, CA, 2011, pp. 2050–2055.

[17] F. Lin, M. Fardad, and M. R. Jovanović, "Design of optimal sparse feedback gains via the alternating direction method of multipliers," *IEEE Trans. Automat. Control*, vol. 58, no. 9, pp. 2426–2431, 2013.

[18] F. Dörfler, M. R. Jovanović, M. Chertkov, and F. Bullo, "Sparse and optimal wide-area damping control in power networks," in *Proceedings of the 2013 American Control Conference*, Washington, DC, 2013, pp. 4295–4300.

[19] F. Dörfler, M. R. Jovanović, M. Chertkov, and F. Bullo, "Sparsity-promoting optimal wide-area control of power networks," *IEEE Trans. Power Syst.*, 2014, doi:10.1109/TPWRS.2014.2304465.

[20] P. Kundur, *Power system stability and control*. McGraw-Hill, 1994.

[21] B. Pal and B. Chaudhuri, *Robust control in power systems*. Springer, 2005.

[22] E. J. Candès, M. B. Wakin, and S. P. Boyd, "Enhancing sparsity by reweighted ℓ_1 minimization," *J. Fourier Anal. Appl*, vol. 14, pp. 877–905, 2008.

[23] M. Grant and S. P. Boyd, "CVX: Matlab software for disciplined convex programming, version 2.0 beta," http://cvxr.com/cvx, 2013.