The Dynamic Mode Decomposition: Extensions and Variations

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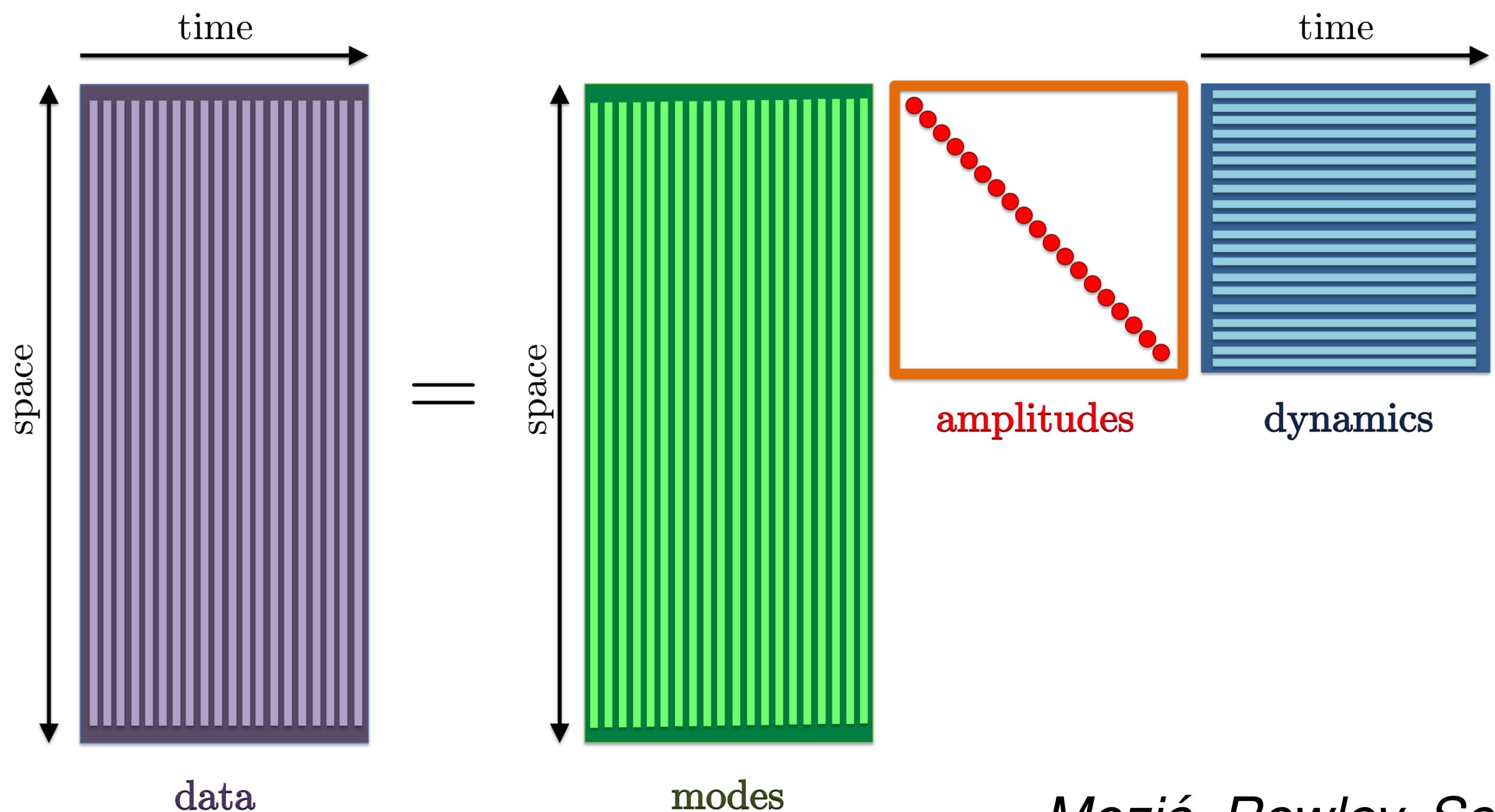
Dynamic Mode Decomposition

- **Represent field of interest as a linear combination of DMD modes**

$$\begin{bmatrix} u_t \end{bmatrix} \approx \sum_{i=1}^{r} \begin{bmatrix} \phi_i \end{bmatrix} a_i \, \lambda_i^t$$



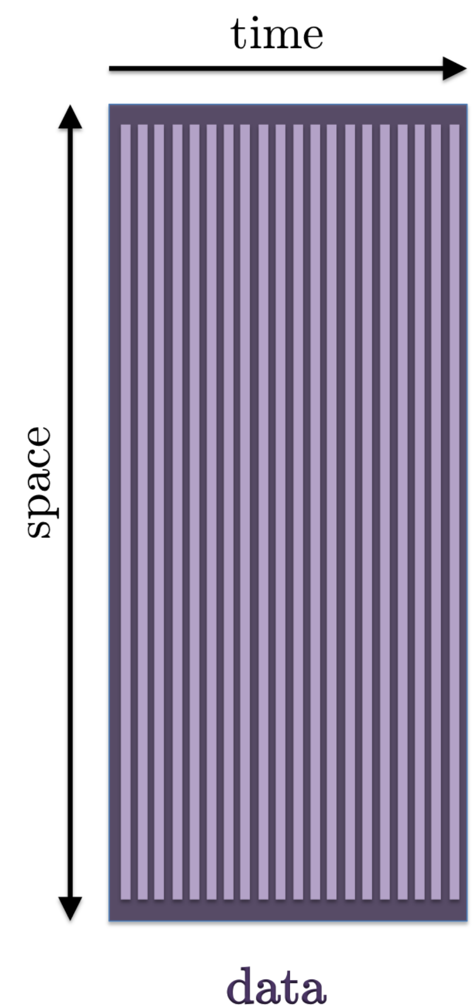
Mezić, Rowley, Schmid

Optimal Amplitudes

- **Least-squares problem for optimal amplitudes** $D_a \;:=\; \mathrm{diag}\,\{a\}$

$$\underset{a}{\text{minimize}} \quad J(a) \;:=\; \|\, G \;-\; L\, D_a\, R\,\|_F^2$$

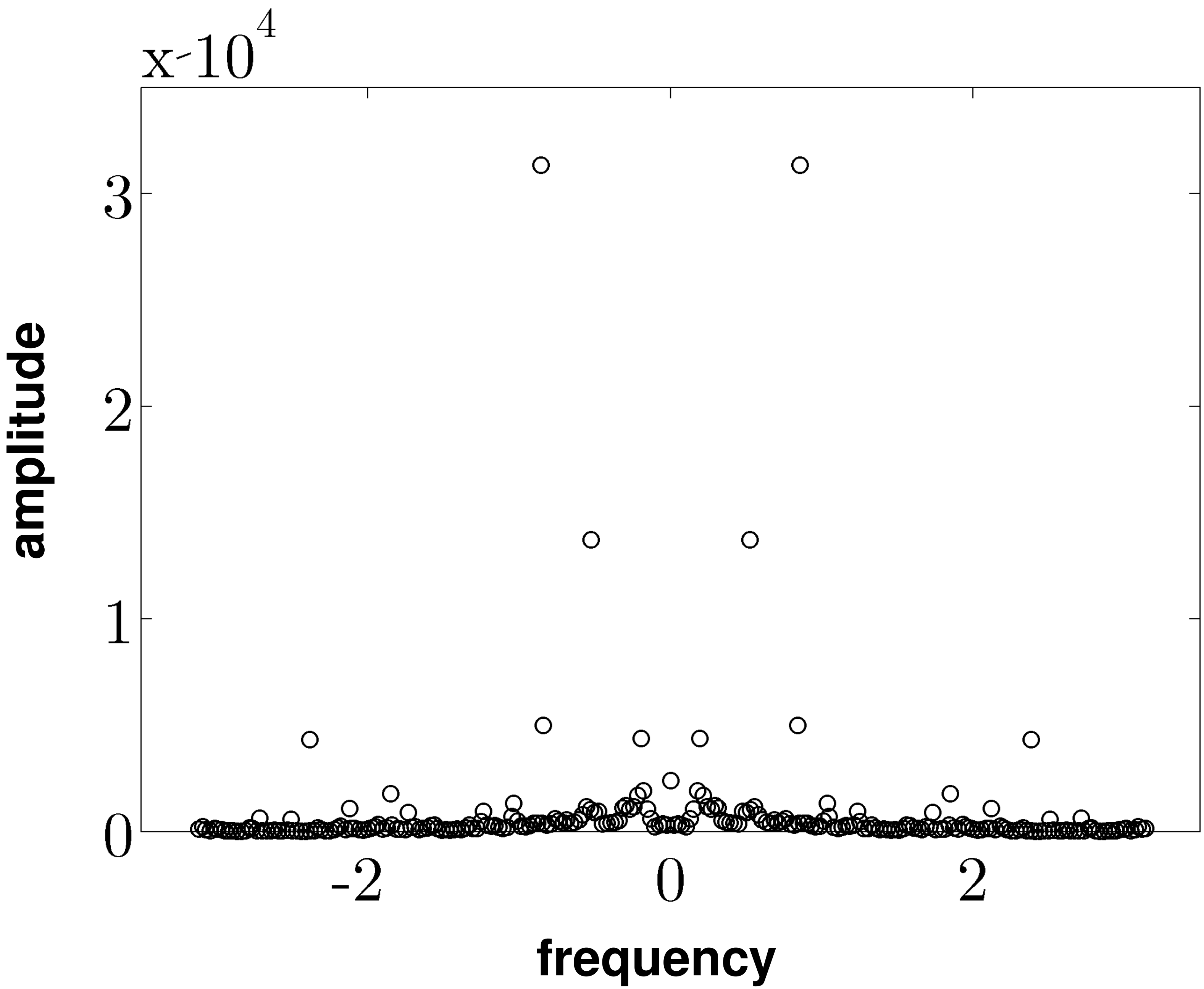
optimal approximation of the matrix of snapshots



Chen, Tu, Rowley, J. Nonlinear Sci. '12

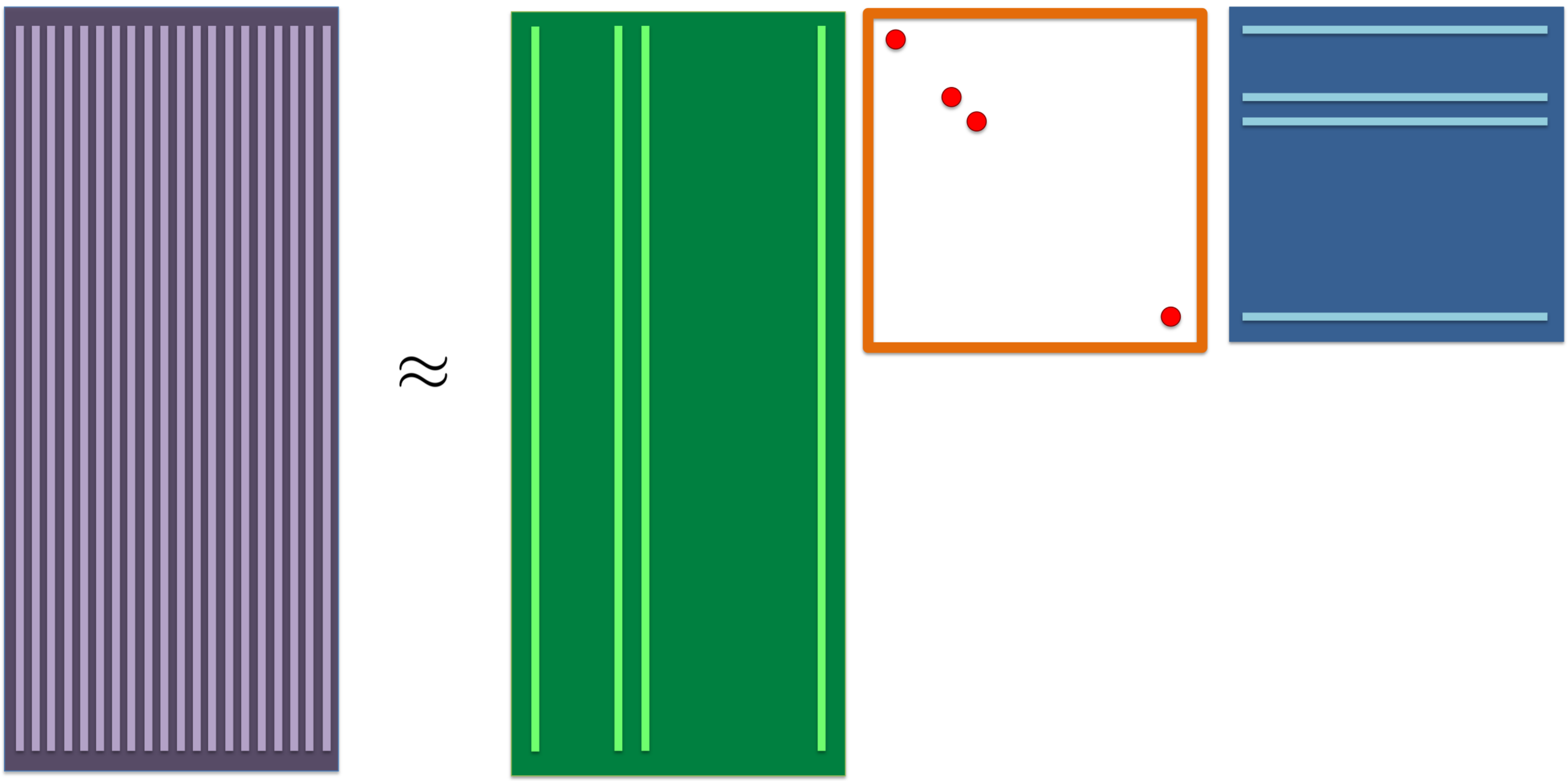
AN EXAMPLE

- **unstructured LES of a screeching supersonic jet**



• Challenge

⋆ **desirable tradeoff** **between**
- **quality of approximation**
- **number of DMD modes**



Outline

❶ SPARSITY-PROMOTING DMD

- ⋆ **Performance vs sparsity**
- ⋆ **Tools from optimization and compressive sensing**

❷ ALGORITHM

- ⋆ **Alternating direction method of multipliers**

❸ AN EXAMPLE

- ⋆ **Screeching supersonic jet**

❹ SUMMARY

Sparsity-promoting DMD

$$\text{minimize} \quad J(a) \quad + \quad \gamma \, \mathbf{card}\,(a)$$

$$\downarrow \qquad\qquad \downarrow$$

least-squares approximation — **sparsity-promoting penalty function**

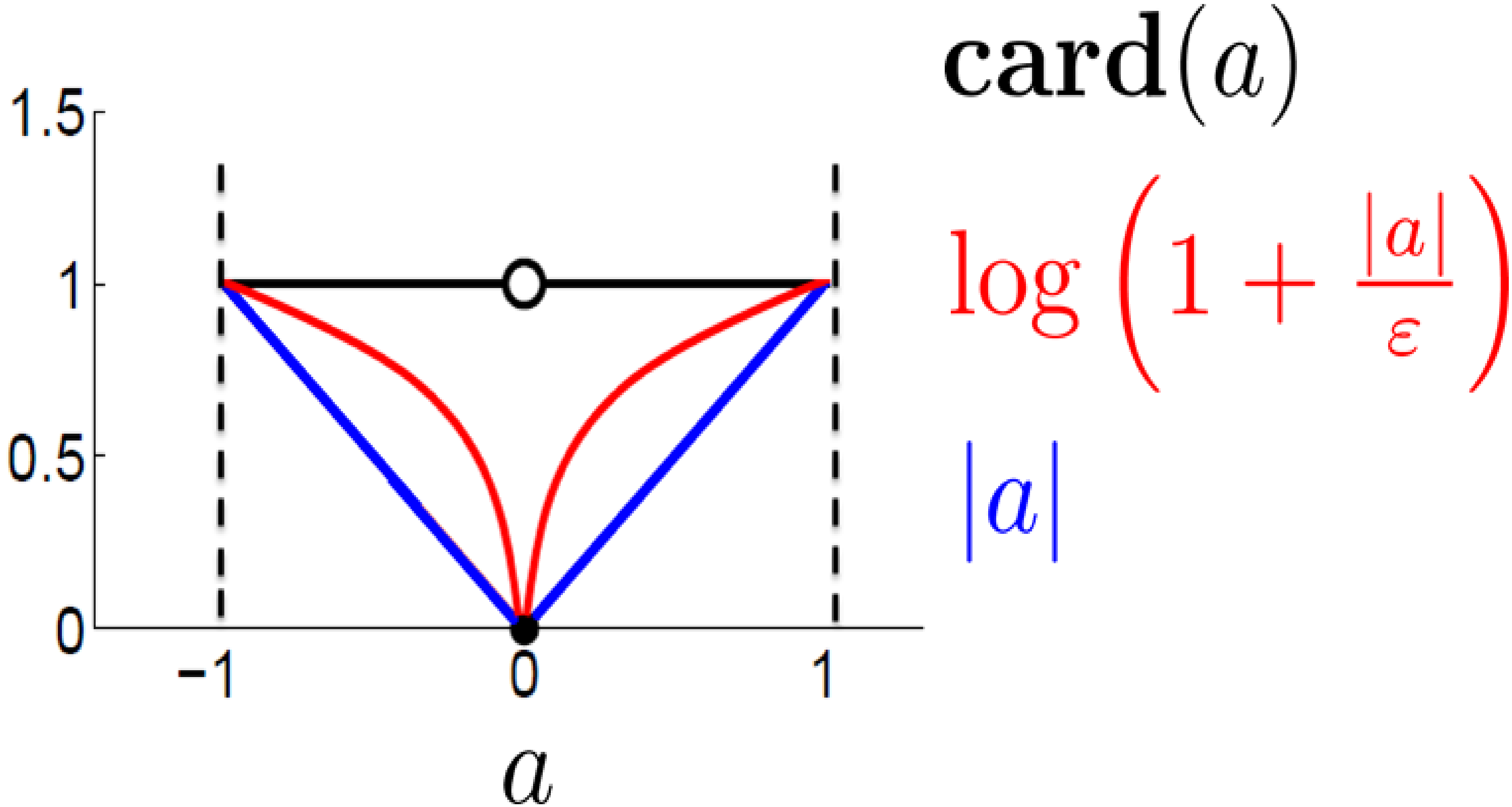
⋆ $\mathbf{card}\,(a)$ – number of non-zero elements

$$a \; := \; \left[\; 1.4 \quad 0 \quad 0 \quad -8.1 \quad 0 \;\right]^T \quad \Rightarrow \quad \mathbf{card}\,(a) \; = \; 2$$

⋆ $\gamma > 0$ – performance vs sparsity tradeoff

Convex Relaxation of $\mathrm{card}\,(a)$

ℓ_1 norm: $\sum_i |a_i|$



Convex Optimization Problem

- **Step 1: structure-identification**

$$\text{minimize} \quad J(a) \quad + \quad \gamma \sum_{i=1}^{r} |a_i|$$

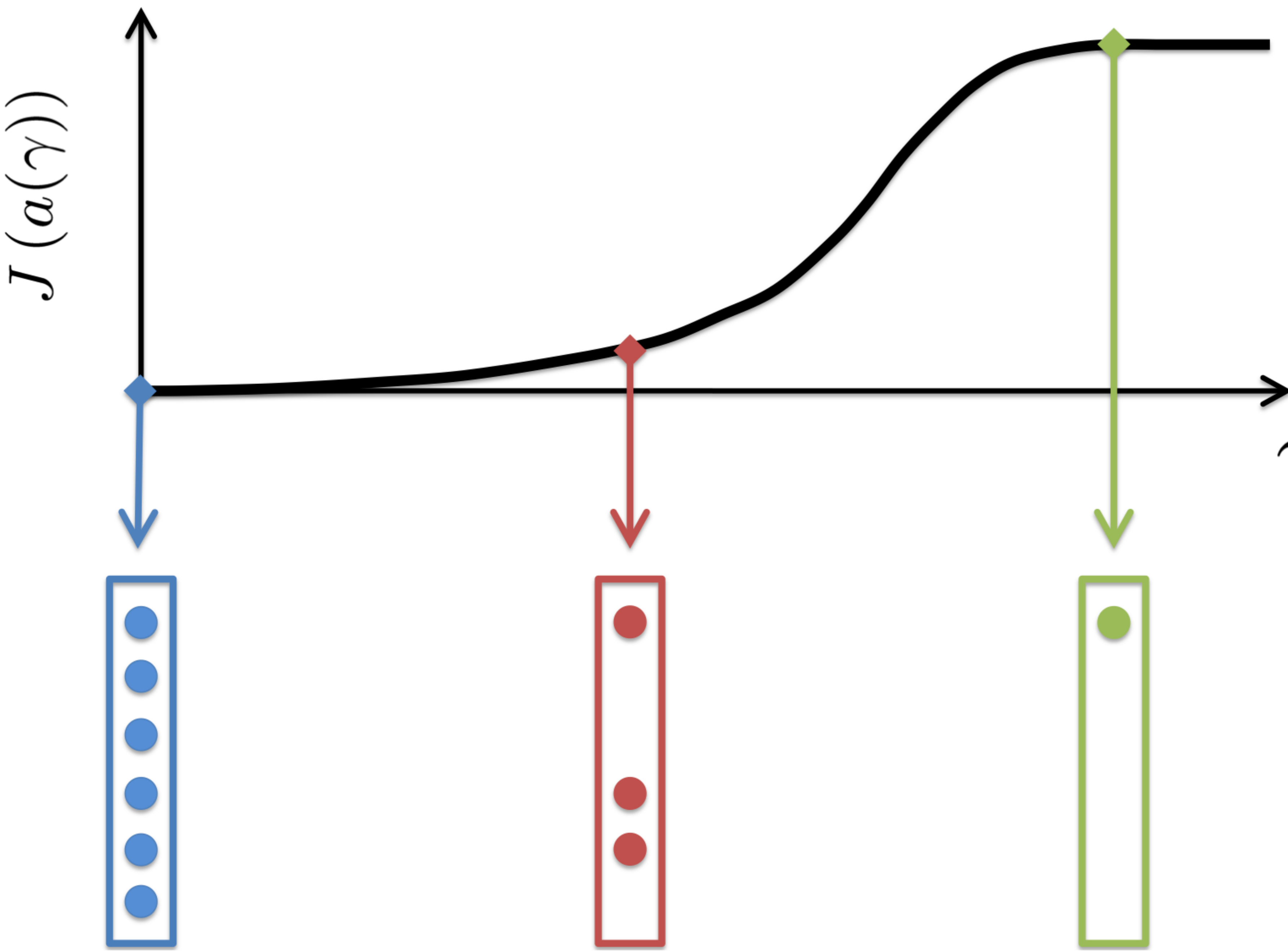
$J(a) \downarrow$ **least-squares approximation**

$\gamma \sum_{i=1}^{r} |a_i| \downarrow$ **proxy for cardinality minimization**

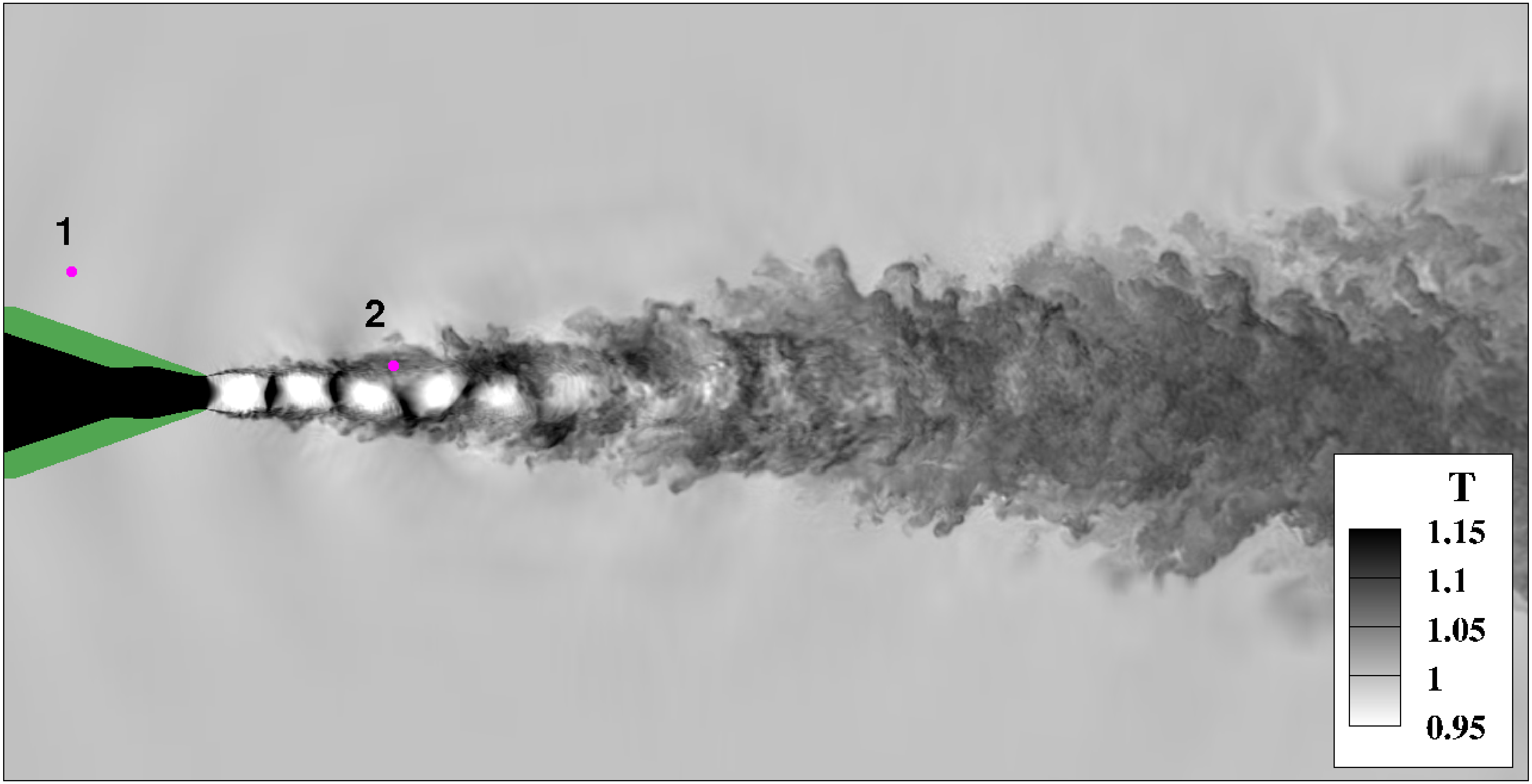
- **Step 2: polishing**

$$\begin{array}{ll} \text{minimize} & J(a) \\ \text{subject to} & E\,a \;=\; 0 \end{array}$$

- **Outcome: parameterized family of optimal amplitudes**

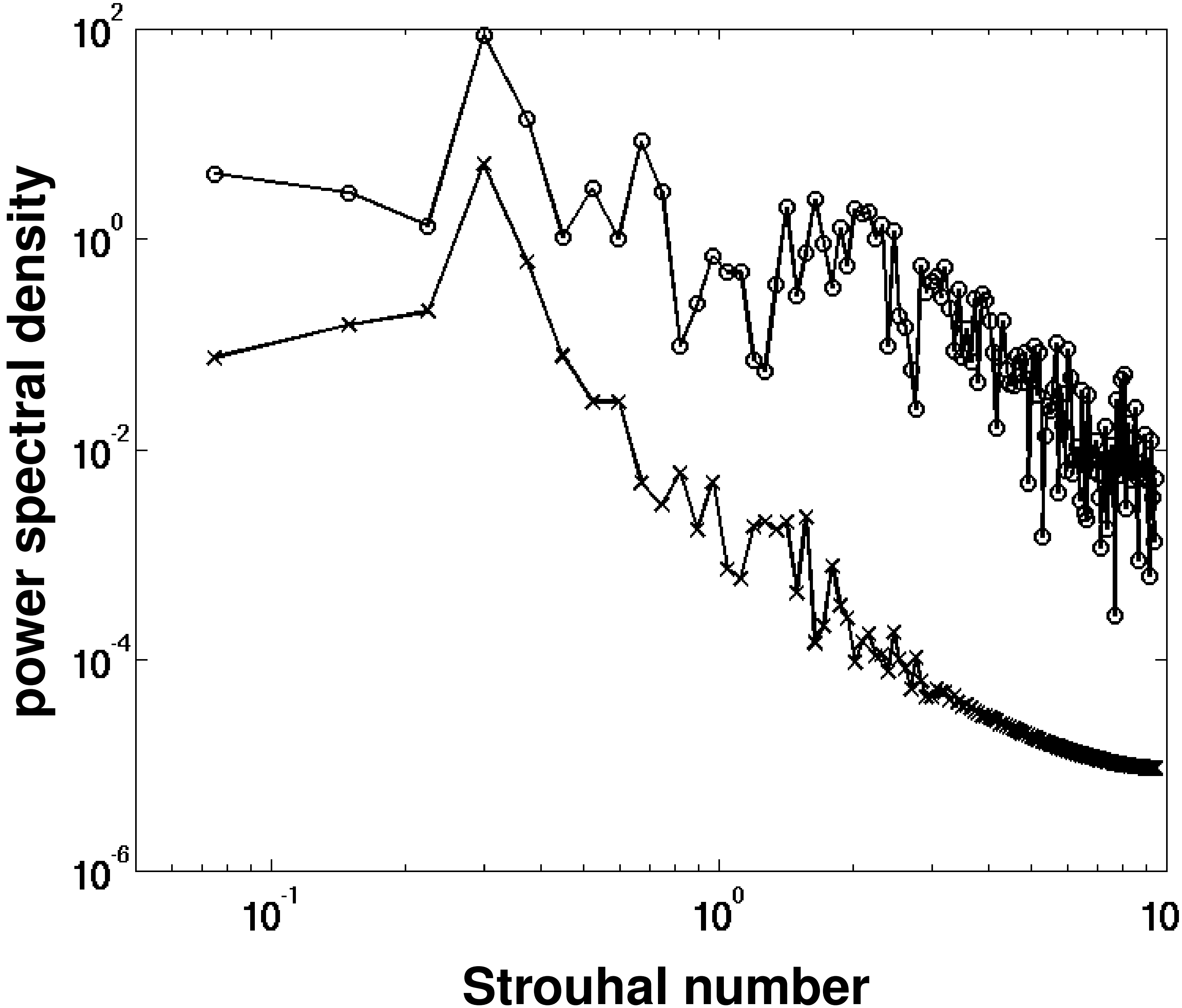


An example: Screeching Supersonic Jet



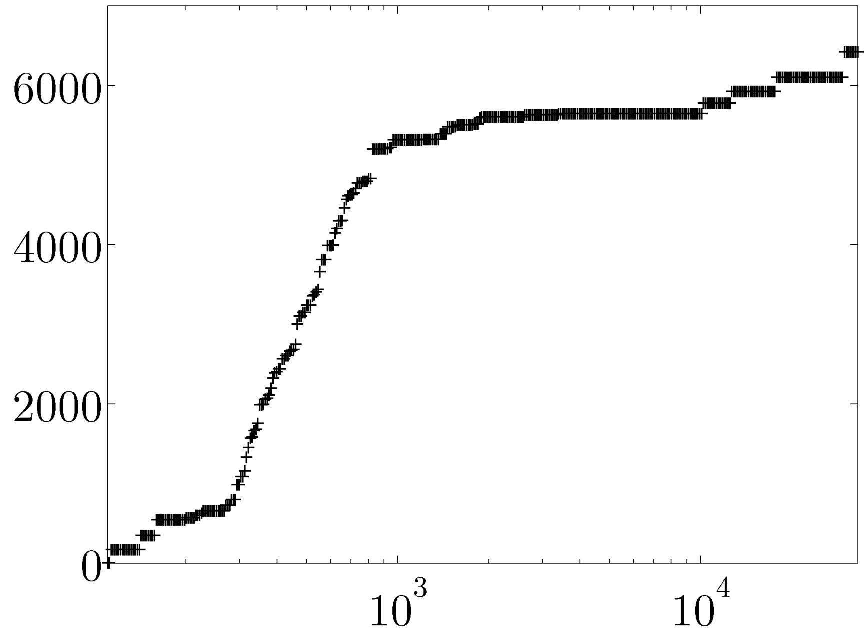
- UNSTRUCTURED LES OF A RECTANGULAR JET
  - ⋆ Aspect ratio 4:1; Mach 1.4
  - ⋆ 45M control volumes (CharLES)

• PRESSURE AT TWO LOCATIONS



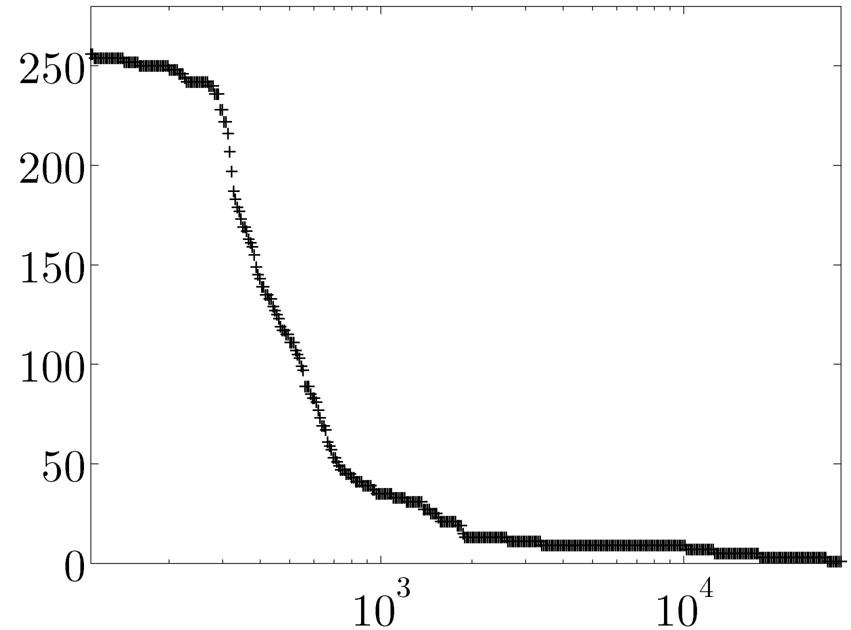
Performance vs Sparsity

least-squares approximation:



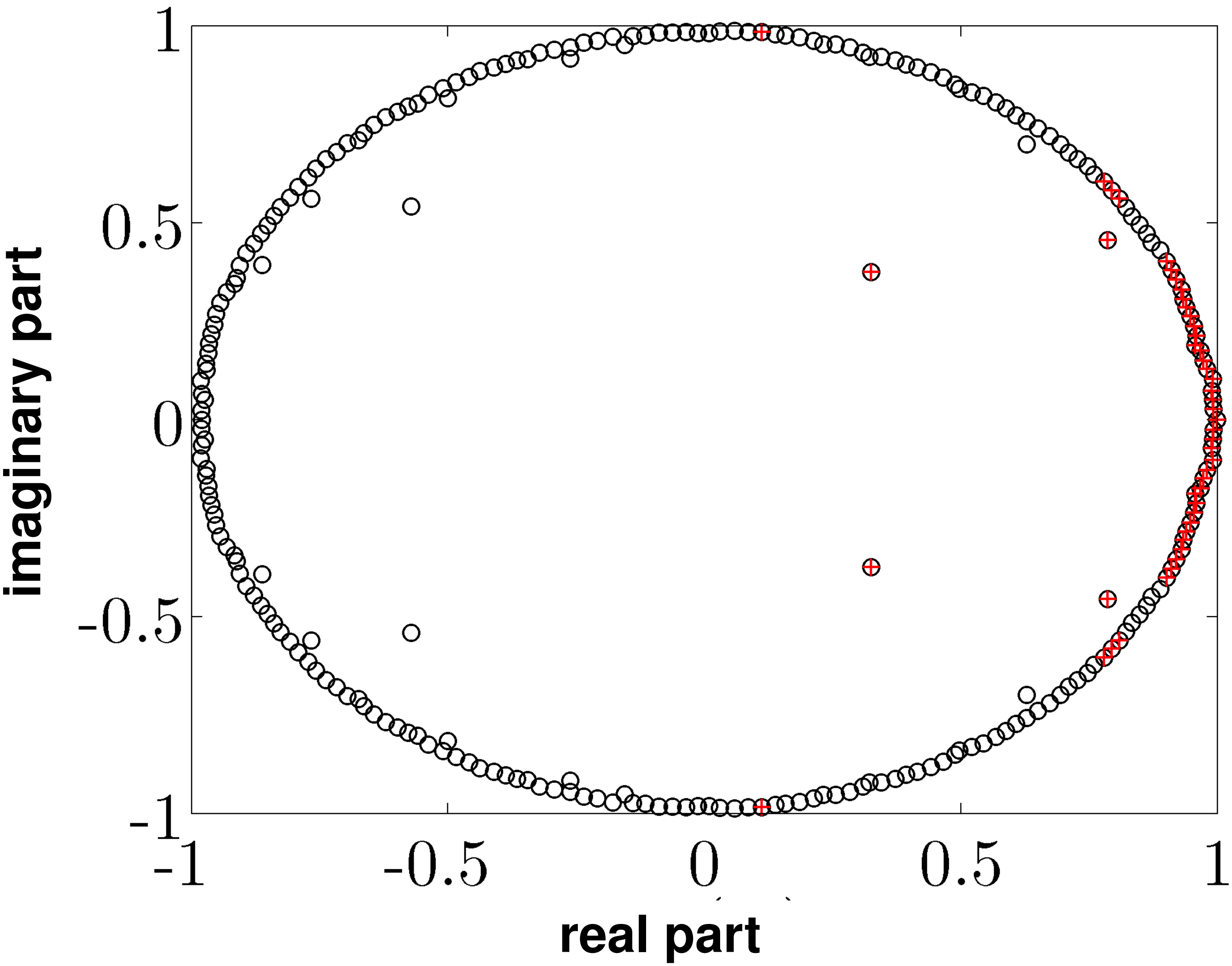
sparsity-promoting parameter

number of non-zero amplitudes:

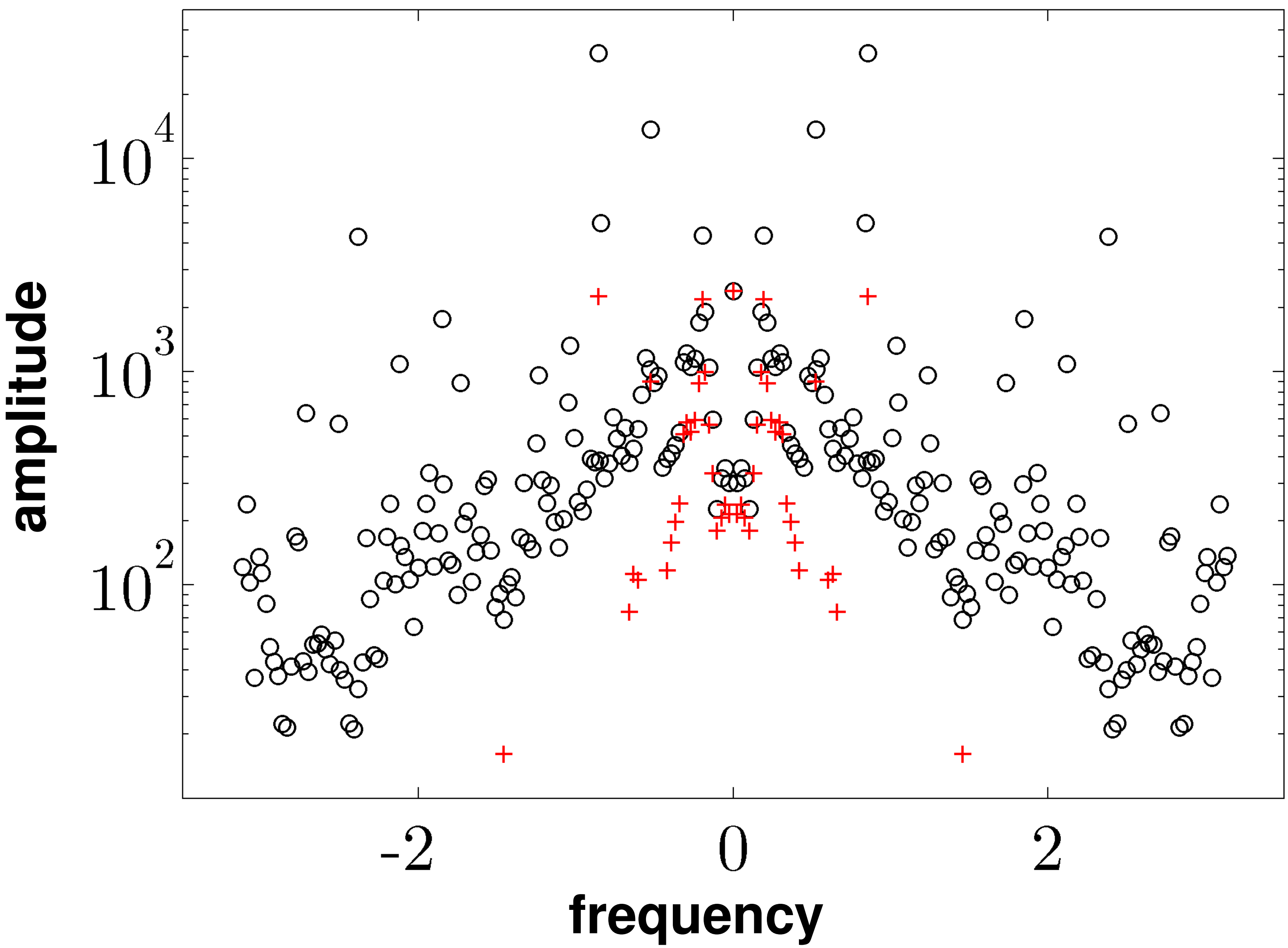


sparsity-promoting parameter

- SPECTRUM
  - $N_z = 47$

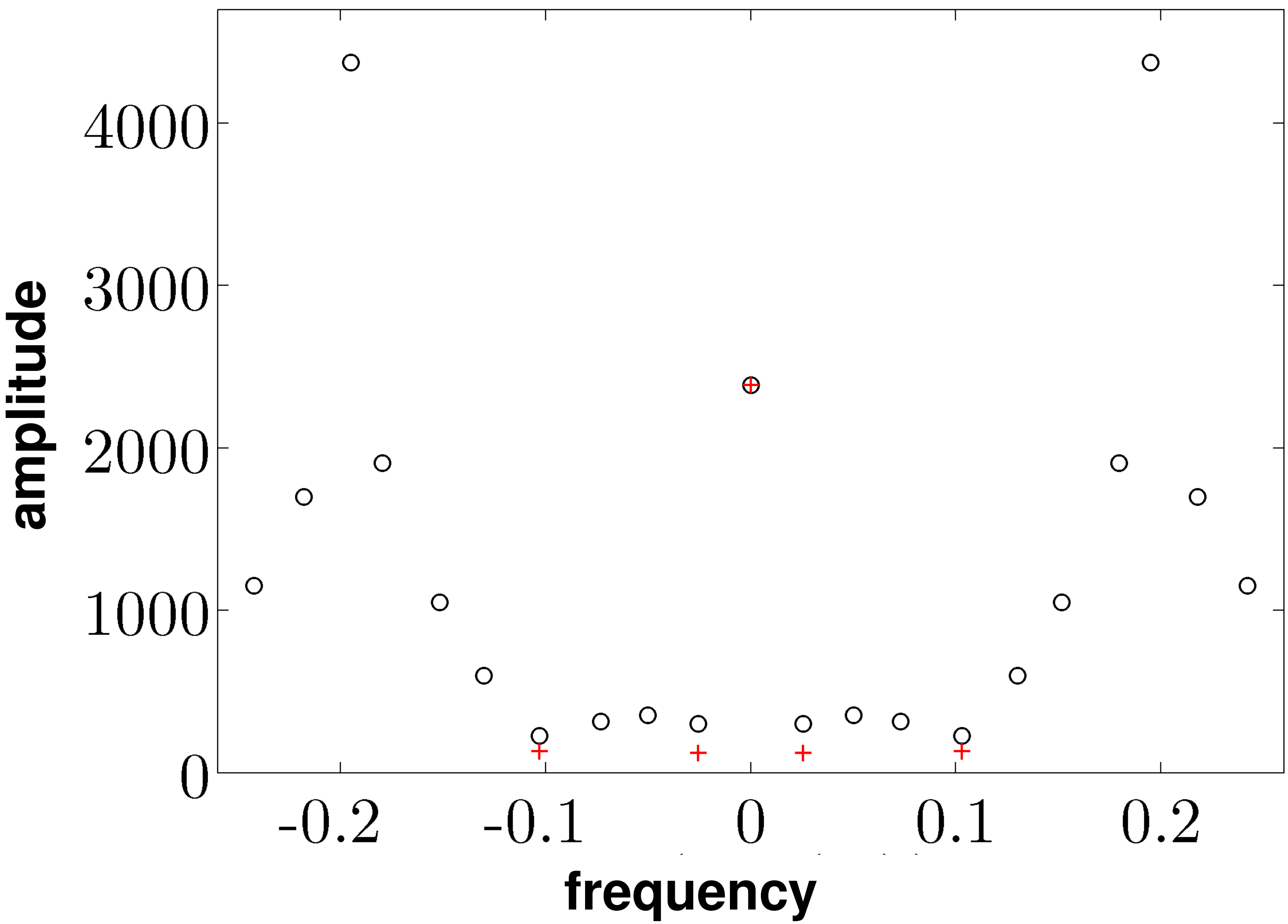


- AMPLITUDE VS FREQUENCY
  - $N_z = 47$



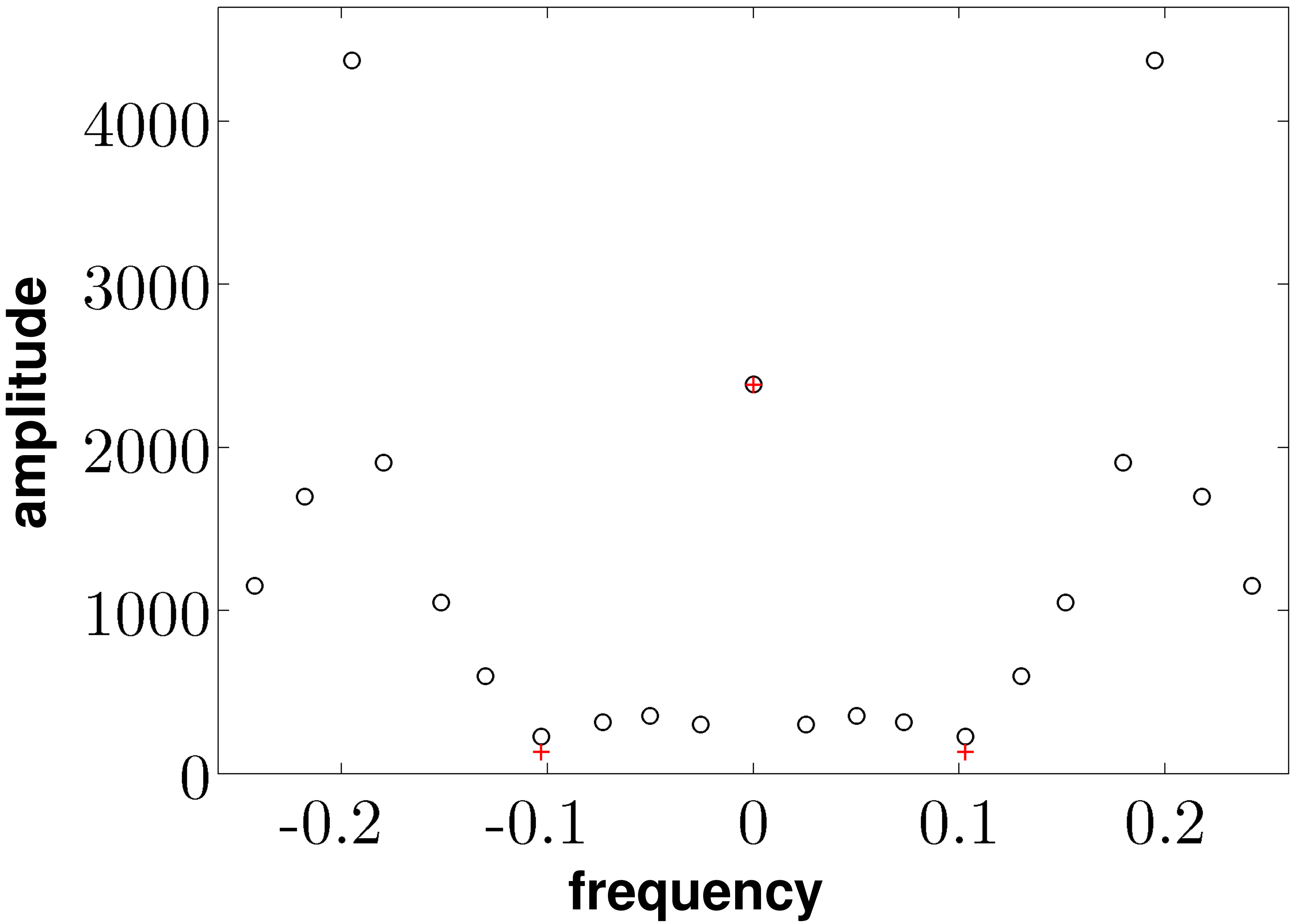
• Amplitude vs frequency

⋆ $N_z = 5$



• AMPLITUDE VS FREQUENCY

⋆ $N_z = 3$



Algorithm: Alternating Direction Method of Multipliers

$$\text{minimize} \quad J(a) \; + \; \gamma \, g(a)$$

- **Step 1: introduce additional variable/constraint**

$$\begin{array}{ll} \text{minimize} & J(a) \; + \; \gamma \, g(b) \\ \text{subject to} & a \; - \; b \; = \; 0 \end{array}$$

benefit: decouples J and g

- **Step 2: introduce augmented Lagrangian**

$$\mathcal{L}_\rho(a, b, \lambda) \; := \; J(a) \; + \; \gamma \, g(b) \; + \; \langle \lambda, a \; - \; b \rangle \; + \; \frac{\rho}{2} \, \| a \; - \; b \|_2^2$$

- **Step 3: use ADMM for augmented Lagrangian minimization**

$$\mathcal{L}_\rho(a, b, \lambda) \;=\; J(a) \;+\; \gamma\, g(b) \;+\; \langle \lambda, a \,-\, b \rangle \;+\; \frac{\rho}{2}\, \|a \,-\, b\|_2^2$$

ADMM:

$$\begin{aligned} a^{k+1} &:= \arg\min_a \mathcal{L}_\rho(a, b^k, \lambda^k) \\ b^{k+1} &:= \arg\min_b \mathcal{L}_\rho(a^{k+1}, b, \lambda^k) \\ \lambda^{k+1} &:= \lambda^k \,+\, \rho\,(a^{k+1} \,-\, b^{k+1}) \end{aligned}$$

- **Step 4: Polishing – structured optimal design**

  ⋆ ADMM: **tool for identifying sparsity patterns**

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REFERENCE

Jovanović, Schmid, Nichols, CTR Annual Research Briefs 2012