Distributed Control: Optimality and Architecture

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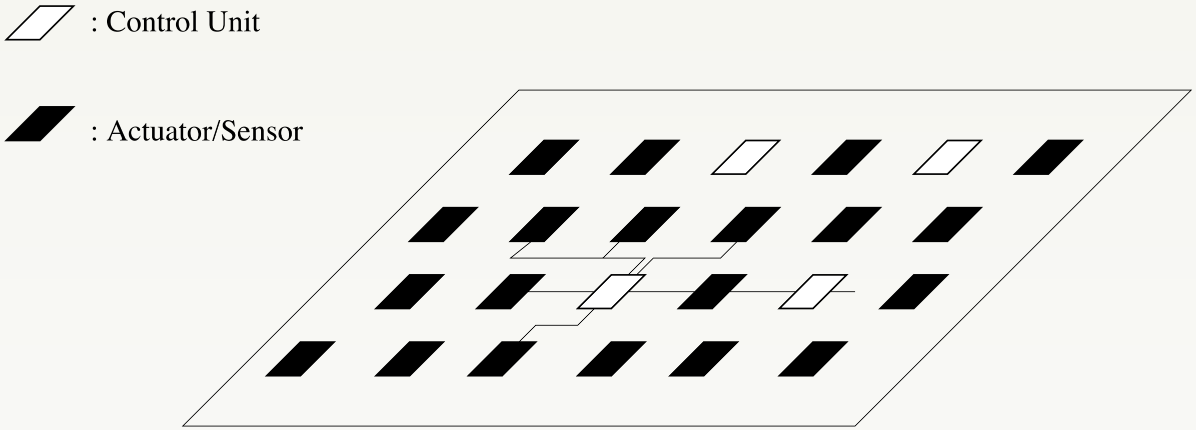
Large Arrayed Systems of Sensors and Actuators

- **New (and old) technologies**
  - Micro-Electro-Mechanical-Systems (MEMS) $\longrightarrow$ Large Arrays
  - Vehicular Platoons
  - Cross Directional (CD) control in pulp and paper processes

- **Modeling and control issues**
  - Complexity (Control-Oriented Modeling)
  - Overall *System Design* (vs. individual device design)
  - Controller architecture

- **Distributed Systems Theory**
  - Infinite-dimensional systems with special structure
  - Controller architecture

Arrays of Micro-Electro-Mechanical-Systems (MEMS)



CURRENTLY FEASIBLE: Very large arrays of MEMS with integrated control circuitry

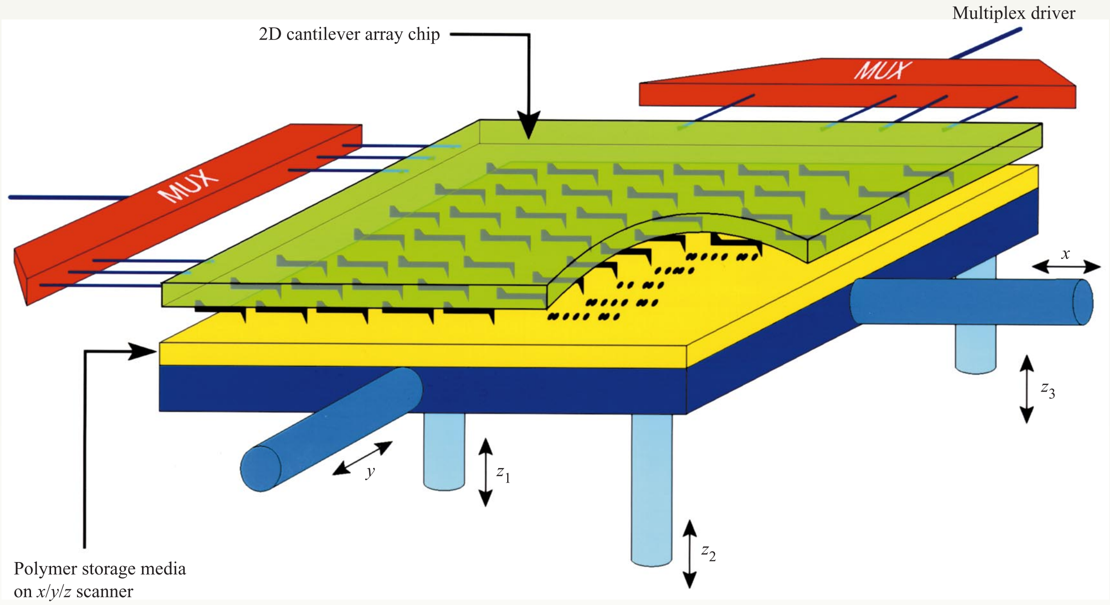
Issues:

- **Tightly coupled dynamics** ⟶ Spatio-temporal instabilities *(e.g. string instability)*

  Current designs avoid this with large spacing

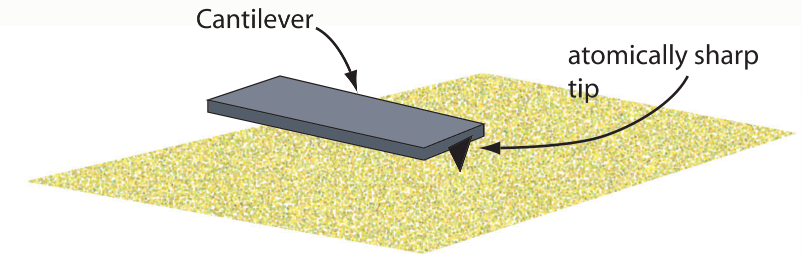
- **Controller architecture**
  - Layout of sensors/actuators
  - Communication between actuators/sensors
    how to decentralize or localize

Example: Massively Parallel Data Storage (IBM Millpede project)



- Atomic level resolution using Atomic Force Microscopy (AFM) and Scanning Tunneling Microscopy (STM) techniques

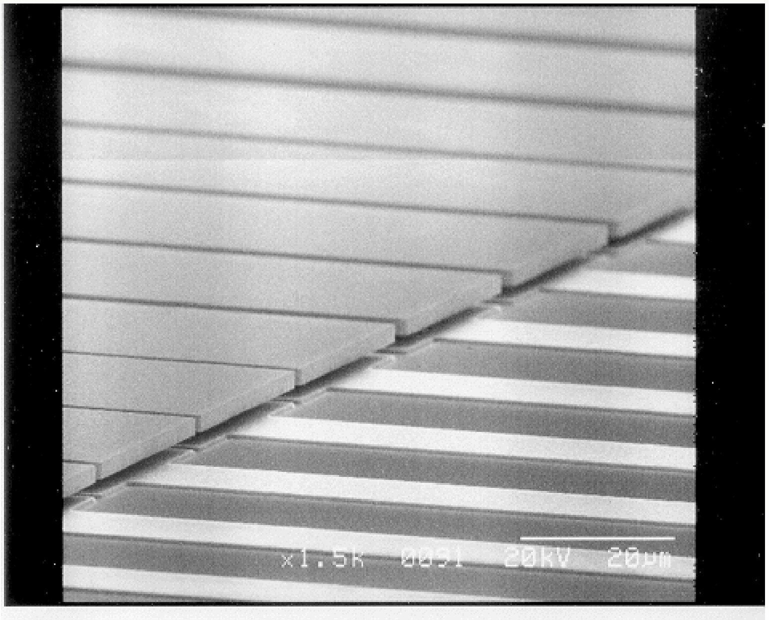
$100 \sim 1000\ Tb/in^2$ density possible!



- **Problem:** slow scans $=$ low throughput
  Solution: go massively parallel

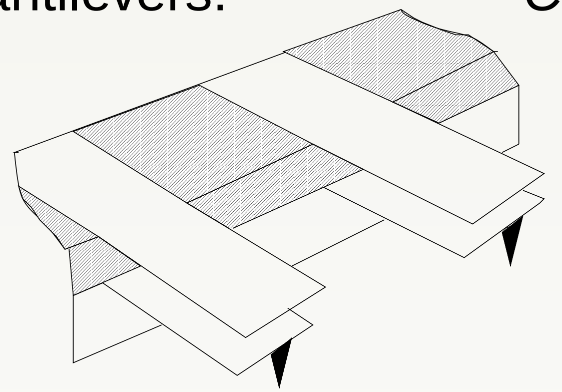
Design and Control Issues in MEMS Arrays

- More tightly packed arrays $\longrightarrow$ more dynamical coupling
  - Micro-cantilever arrays
  - Micro-mirror arrays
- Current fixes:
  - Large spacings
  - Complex design to isolate elements
- Experimental effort at UCSB:
  design deliberately coupled arrays
- Demonstrate “electronic” decoupling
  using feedback



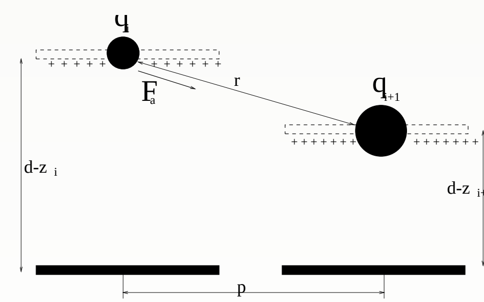
Micro-cantilever Array Control

Capacitively actuated micro-cantilevers: *Combined* actuator and sensor



Important Considerations:

- Higher throughput, faster "access time" $\longrightarrow$ Tightly packed cantilevers
- For tightly packed cantilevers, significant dynamical coupling due to
  - Mechanical coupling
  - Fringe fields
    (Napoli & Bamieh, '01)



- Large arrays $\approx$ 10,000 devices
  $\Rightarrow$ must use localized control

Distributed Systems with Special Structure

- General Infinite-dimensional Systems Theory
  - Well posedness issues (semi-group theory)
  - Constructive (convergent) approximation techniques

THEME: *Make infinite-dimensional problems look like finite-dimensional ones*

- Special Structure
  - Distributed control and measurement *(now more feasible)*
  - Regular (lattice) arrangement of devices

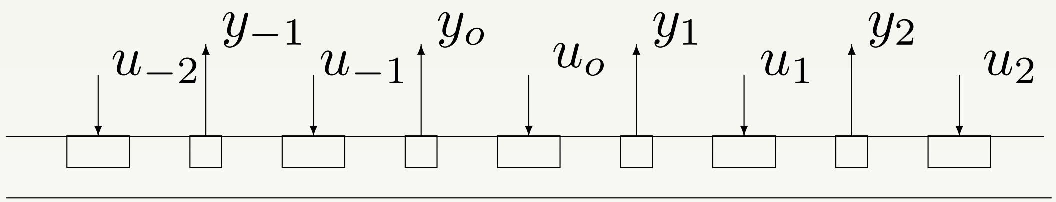
Together $\Longrightarrow$ *Spatial Invariance*

  - Control of "Vehicular Strings", (Melzer & Kuo, 71)
  - Discretized PDEs, (Brockett, Willems, Krishnaprasd, El-Sayed, '74, '81)
  - "Systems over rings", (Kamen, Khargonekar, Sontag, Tannenbaum, ...)
  - Systems with "Dynamical Symmetry", (Fagniani & Willems)

More recently:

  - Controller architecture and localization, (Bamieh, Paganini, Dahleh)
  - LMI techniques, localization, (D'Andrea, Dullerud, Lall)

Example: Distributed Control of the Heat Equation



u_i: input to heating elements. y_i: signal from temperature sensor.
Dynamics are given by:

$$\begin{bmatrix} \vdots \\ y_{-1} \\ y_o \\ y_1 \\ \vdots \end{bmatrix} = \begin{bmatrix} & \vdots & & \vdots & \\ \cdots & & H_{-1,0} & & \cdots \\ & H_{0,-1} & H_{0,0} & H_{0,1} & \\ \cdots & & H_{1,0} & & \cdots \\ & \vdots & & \vdots & \end{bmatrix} \begin{bmatrix} \vdots \\ u_{-1} \\ u_o \\ u_1 \\ \vdots \end{bmatrix}$$

Each $H_{i,j}$ is an infinite-dimensional SISO system.

Fact: Dynamics are spatially invariant $\Rightarrow$ H is Toeplitz

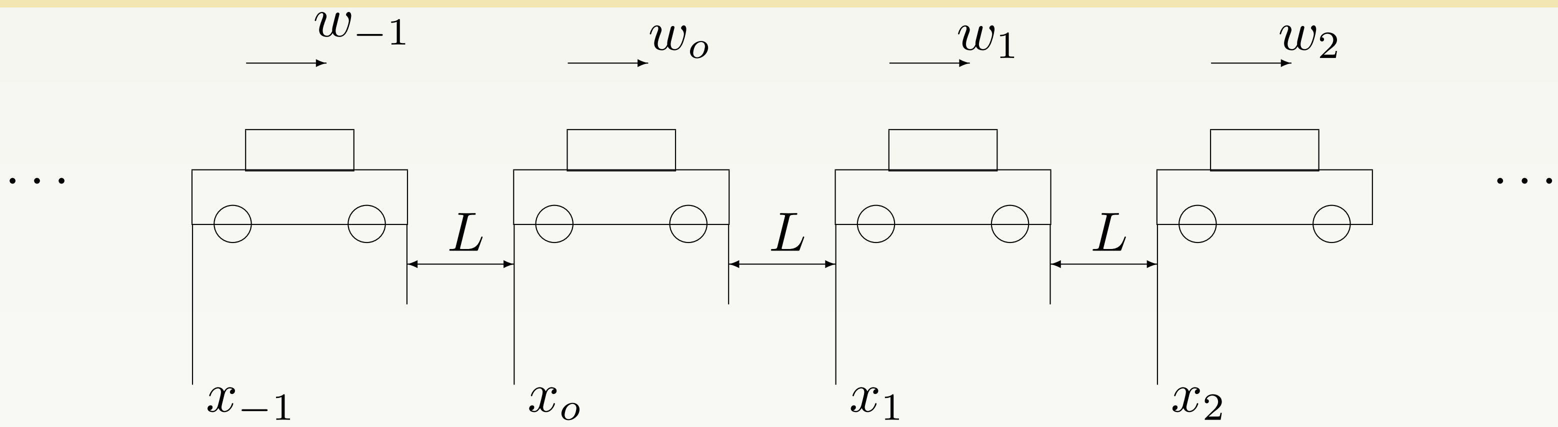
The input-output relation can be written as a *convolution over the actuator/sensor index*:

$$y_i = \sum_{j=-\infty}^{\infty} \bar{H}_{(i-j)} \, u_j,$$

The limit of large actuator sensor array:

$$\frac{\partial \psi}{\partial t}(x,t) \;=\; c\frac{\partial^2 \psi}{\partial x^2}(x,t) \;+\; u(x,t) \qquad\qquad \psi_x \;=\; \int_{-\infty}^{\infty} H_{x-\zeta}\, u_\zeta d\zeta,$$

Vehicular Platoons



Objective: Design a controller for each vehicle to:

- Maintain constant small slot length L.
- Reject the effect of disturbances $\{w_i\}$ (wind gusts, road conditions, etc...)

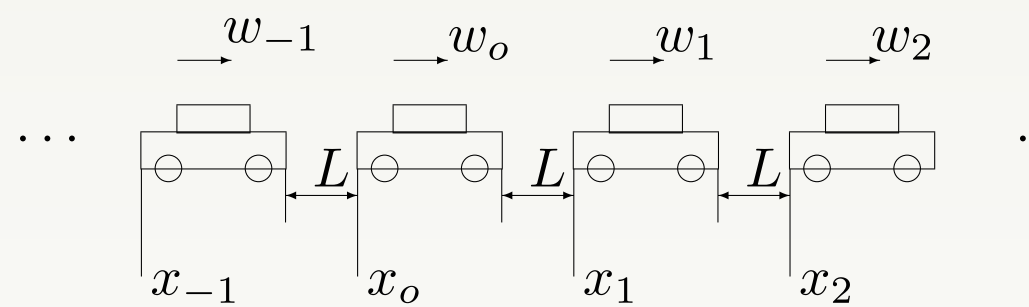
Warning: Designs based on two vehicle models may lack “string stability”, i.e. disturbances get amplified as they propagate through the platoon.

Problem Structure:

- Actuators: each vehicle’s throttle input.
- Sensors: position and velocity of each vehicle.

Vehicular Platoons Set-up

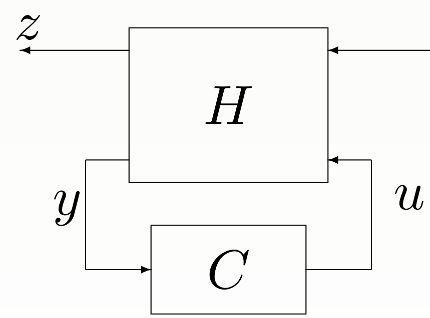
x_i: i'th vehicle's position.



$$
\begin{aligned}
\tilde{x}_i &:= x_i - x_{i-1} - L - C \\
\tilde{x}_{1,i} &:= \tilde{x}_i \\
\tilde{x}_{2,i} &:= \dot{\tilde{x}}_i
\end{aligned}
$$

Structure of generalized plant:

$$
H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} = \begin{bmatrix} \times & & \times & \\ & \ddots & & 0 \\ & \ddots & h_o & \\ \times & & h_1 & \ddots \\ & 0 & & \ddots \end{bmatrix}
$$



The generalized plant has a Toeplitz structure!

$$z = \mathcal{F}(H, C)$$

Optimal Controller for Vehicular Platoon

Example: Centralized $\mathcal{H}^2$ optimal controller gains for a 50 vehicle platoon
(From: Shu and Bamieh '96)

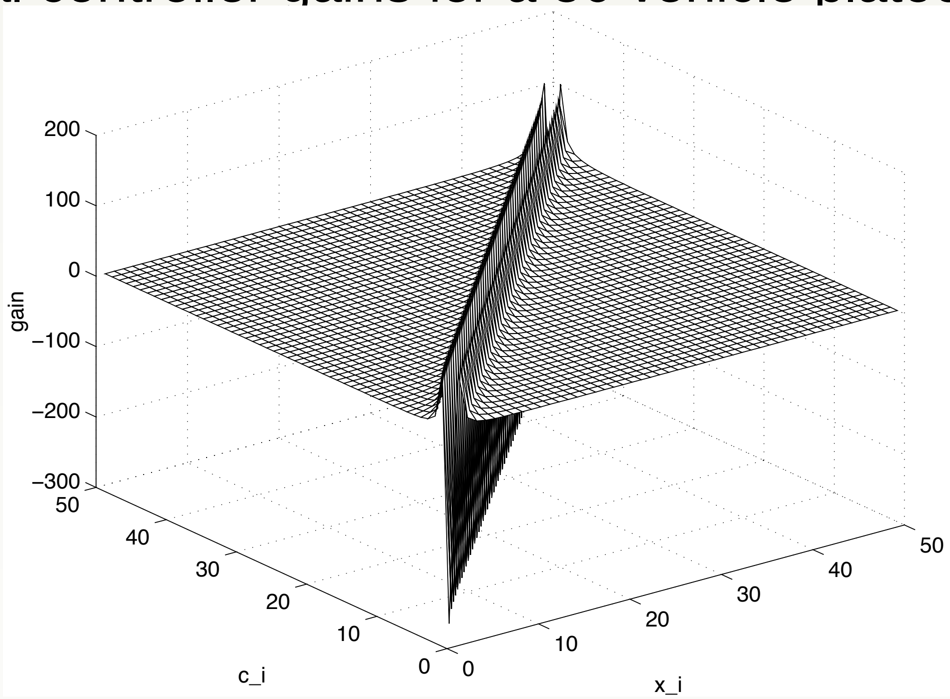


Figure 1: Position error feedback gains for a 50 vehicle platoon

Remarks:

- For large platoons, optimal controller is approximately Toeplitz
- Optimal centralized controller has some inherent decentralization ("localization")
  Controller gains decay away from the diagonal

Q: Do the above 2 results occur in all "such" problems?

Spatial Invariance of Dynamics

Indexing of actuator and sensor signals:

$$u_i(t) \;:=\; u_{(i_1,\ldots,i_n)}(t), \qquad\qquad y_i(t) \;:=\; y_{(i_1,\ldots,i_n)}(t).$$

$i \;:=\; (i_1,\ldots,i_n)$ a spatial multi-index, $\qquad i \in \mathbb{G} \;:=\; \mathbb{G}_1 \times \ldots \times \mathbb{G}_n$.

Linear input-output relations: A general linear system from u to y:

$$y_i \;=\; \sum_{j\in\mathbb{G}} H_{i,j}\, u_j, \qquad \Leftrightarrow \qquad y_{(i_1,\ldots,i_n)} \;=\; \sum_{j_1\in\mathbb{G}_1} \cdots \sum_{j_n\in\mathbb{G}_n} H_{(i_1,\ldots,i_n),(j_1,\ldots,j_n)}\, u_{(j_1,\ldots,j_n)},$$

Spatial Invariance:

Assumption 1: Set of spatial indices $=$ commutative group

$\mathbb{G} \;:=\; \mathbb{G}_1 \times \ldots \times \mathbb{G}_n$, $\qquad$ each $\mathbb{G}_i$ a commutative group.

Remark: “spatial shifting” of signals

$(S_\sigma u)_i \;:=\; u_{i-\sigma}$ $\qquad$ Compare with: *Time shift by* τ $\;(S_\tau u)(t) := u(t-\tau)$

Assumption 2: Spatial invariance $\longleftrightarrow$ Commute with spatial shifts

$$\forall \sigma \in \mathbb{G}, \qquad H\ S_\sigma = S_\sigma\ H \qquad \Leftrightarrow \qquad S_\sigma^{-1} H S_\sigma = H$$

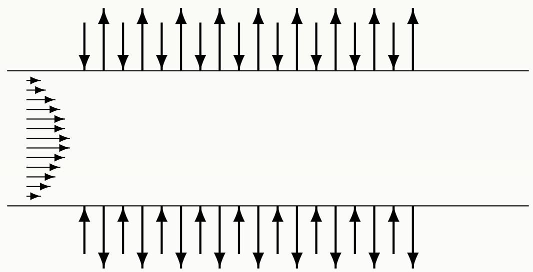
Examples of Spatial Invariance

Generally: Spatial invariance easily ascertained from basic physical symmetry!

- Vehicular platoons: signals index over $\mathbb{Z}$.

- Channel flow: Signals indexed over $\{0,1\} \times \mathbb{Z}$:

$$y_{(l,i)} \;=\; \sum_{j=-\infty}^{\infty} H_{(l-0,i-j)}\, u_{(0,j)} \;+\; \sum_{j=-\infty}^{\infty} H_{(l-1,i-j)}\, u_{(1,j)}, \qquad l \;=\; 0,1.$$



Remark: The input-output mapping of a spatially invariant system can be rewritten:

$$y_i \;=\; \sum_{j \in \mathbb{G}} \bar{G}_{i-j}\, u_j, \qquad \Leftrightarrow \qquad y_{(i_1,\ldots,i_n)} \;=\; \sum_{j_1 \in \mathbb{G}_1} \cdots \sum_{j_n \in \mathbb{G}_n} \bar{G}_{(i_1-j_1,\ldots,i_n-j_n)}\, u_{(j_1,\ldots,j_n)}.$$

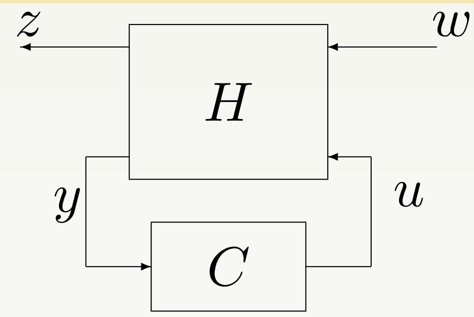
A spatial convolution

Symmetry in Dynamical Systems and Control Design

- Many-body systems always have some inherent dynamical symmetries:
  e.g. equations of motion are invariant to certain coordinate transformations
- **Question:** Given an unstable dynamical system with a certain symmetry,
  is it possible to stabilize it with a controller that has the same symmetry?
  (i.e. without "breaking the symmetry")
- **Answer:** Yes! (Fagnani & Willems '93)

Remark: Spatial invariance is a dynamical symmetry
This answer applies to optimal design as well

i.e.
For best achievable performance, need only consider spatially-invariant controllers



$$z = \mathcal{F}(H,C)\, w$$

The standard problem:

Signal norms:

$$\|w\|_p^p := \sum_{i\in\mathbb{G}} \int_{\mathbb{R}} |w_i(t)|^p dt = \sum_{i\in\mathbb{G}} \|w\|_p^p$$

Induced system norms:

$$\|\mathcal{F}(G,C)\|_{p-i} := \sup_{w\in L^P} \frac{\|z\|_p}{\|w\|_p}.$$

The $\mathcal{H}^2$ norm:

$$\|\mathcal{F}(G,C)\|_{\mathcal{H}^2}^2 = \|z\|_2^2 = \sum_{i\in\mathbb{G}} \|z_i\|_{L^2}^2,$$

with impulsive disturbance input $w_i(t) = \delta(i)\delta(t)$.

Note: In the platoon problem: finite system norm $\Rightarrow$ string stability.

Spatially-Invariant vs. Spatially-Varying Controllers

Question: Are spatially-varying controllers better than spatially-invariant ones?

Answer: If plant is spatially invariant, no!

$LSI :=$ The class of Linear Spatially-Invariant systems.

$LSV :=$ The class of Linear Spatially-Varying systems.

Compare the two problems:

$$\gamma_{si} := \inf_{\substack{\text{stabilizing } C \\ C \in LSI}} \|\mathcal{F}(G,C)\|_{p-i}$$

The best achievable performance with spatially-invariant controllers

$$\gamma_{sv} := \inf_{\substack{\text{stabilizing } C \\ C \in LSV}} \|\mathcal{F}(G,C)\|_{p-i}$$

The best achievable performance with spatially-varying controllers

Theorem 1. *If the plant and performance objectives are spatially invariant, i.e. if the generalized plant G is spatially invariant, then the best achievable performance can be approached with a spatially invariant controller. More precisely*

$$\gamma_{si} = \gamma_{sv}.$$

Spatially-Invariant vs. Spatially-Varying Controllers (Cont.)

Related Problem: *Time-Varying vs. Time-Invariant Controllers*

Fact: For time-invariant plants, time-varying controllers offer no advantage over time-invariant ones! *for norm minimization problems*

Proofs based on use of YJBK parameterization. Convert to

$$\gamma_{ti} := \inf_{\substack{\text{stable } Q \\ Q \in LTI}} \|T_1 - T_2 Q T_3\| \qquad \gamma_{tv} := \inf_{\substack{\text{stable } Q \\ Q \in LTV}} \|T_1 - T_2 Q T_3\| ,$$

T_1, T_2, T_3 determined by plant, therefore time invariant.

Spatially-Invariant vs. Spatially-Varying Controllers (Cont.)

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T_1, T_2, T_3 determined by plant, therefore time invariant.

- The $\mathcal{H}^\infty$ case: (Feintuch & Francis, '85), (Khargonekar, Poolla, & Tannenbaum, '85). *A consequence of Nehari's theorem*
- The ℓ^1 case: (Shamma & Dahleh, '91). *Using an averaging technique*
- Any induced ℓ^p norm: (Chapellat & Dahleh, '92). *Generalization of the averaging technique*

Spatially-Invariant vs. Spatially-Varying Controllers (Cont.)

Idea of proof: After YJBK parameterization:

$$\gamma_{si} := \inf_{\substack{\text{stable } Q \\ Q \in LSI}} \|T_1 - T_2 Q T_3\| \geq \gamma_{sv} := \inf_{\substack{\text{stable } Q \\ Q \in LSV}} \|T_1 - T_2 Q T_3\|$$

Let $\bar{Q}$ achieve a performance level $\bar{\gamma} = \|T_1 - T_2 \bar{Q} T_3\|$.

Averaging $\bar{Q}$:

- If $\mathbb{G}$ is finite: define

$$Q_{av} := \frac{1}{|\mathbb{G}|} \sum_{\sigma \in \mathbb{G}} \sigma^{-1} \bar{Q} \sigma. \rightarrow Q_{av} \textit{ is spatially invariant, i.e. } \forall \sigma \in \mathbb{G}, \quad \sigma^{-1} Q_{av}\, \sigma = Q_{av}$$

Then

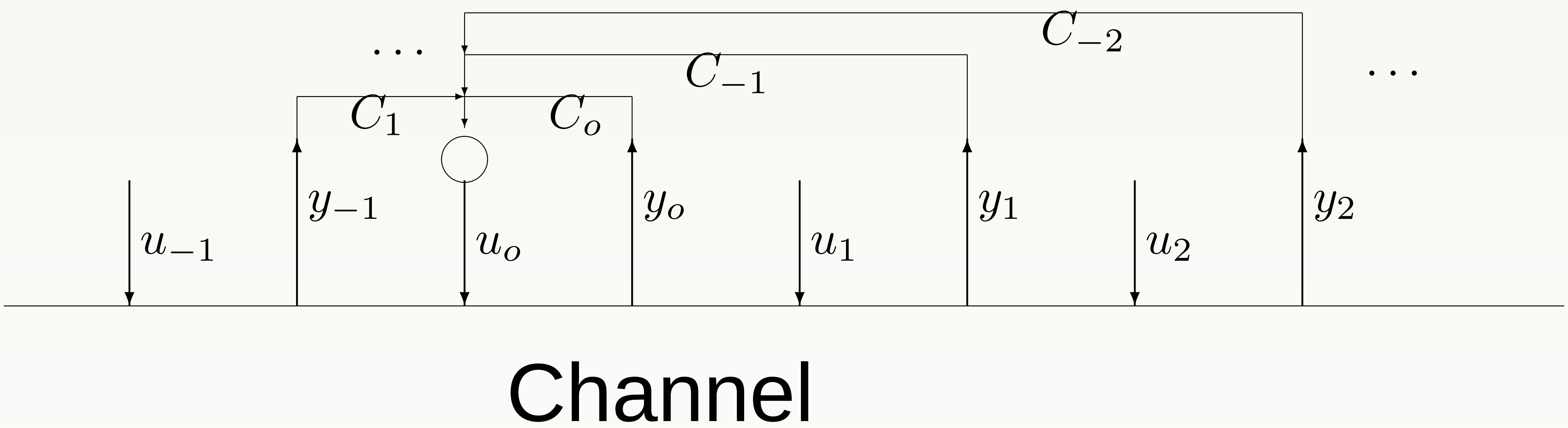
$$\begin{aligned} \|T_1 - T_2 Q_{av} T_3\| &= \left\|T_1 - T_2 \left(\frac{1}{|\mathbb{G}|} \sum_{\sigma \in \mathbb{G}} \sigma^{-1} \bar{Q} \sigma \right) T_3\right\| = \left\| \frac{1}{|\mathbb{G}|} \sum_{\sigma \in \mathbb{G}} \sigma^{-1} \left(T_1 - T_2 \bar{Q} T_3\right) \sigma \right\| \\ &\leq \frac{1}{|\mathbb{G}|} \sum_{\sigma \in \mathbb{G}} \left\| \sigma^{-1} \left(T_1 - T_2 \bar{Q} T_3\right) \sigma \right\| = \|T_1 - T_2 \bar{Q} T_3\| \end{aligned}$$

- If $\mathbb{G}$ is infinite, take a sequence of finite subsets $M_1 \subset M_2 \subset \cdots$, with $\bigcup_n M_n = \mathbb{G}$,

Then define: $$Q_n := \frac{1}{|M_n|} \sum_{\sigma \in M_n} \sigma^{-1} \bar{Q} \sigma.$$

Q_n converges weak $*$ to a spatially-invariant Q_{av} with the required norm bound.

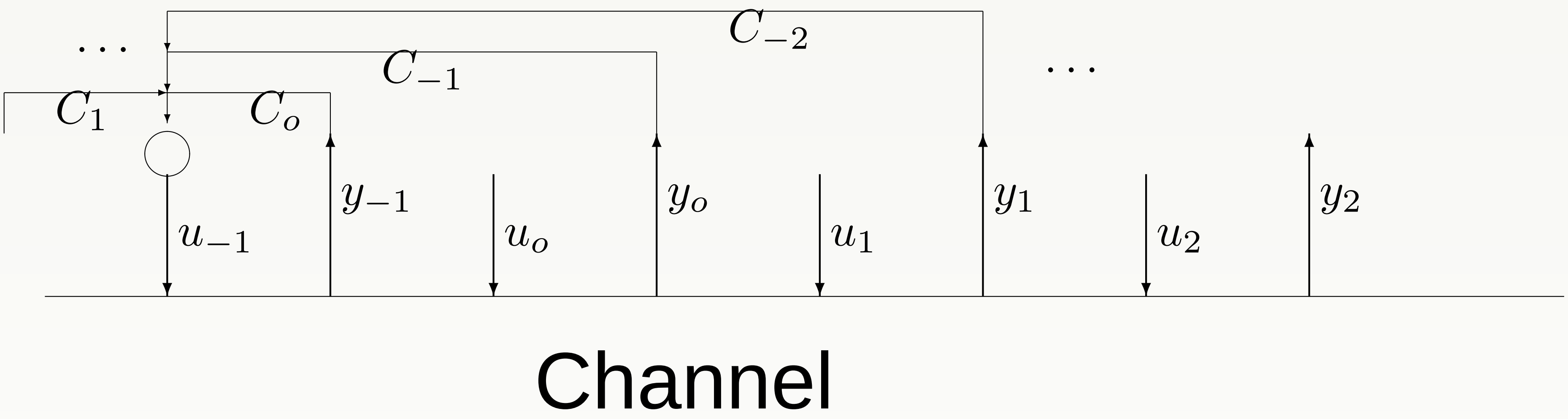
Poiseuille flow stabilization:



$$u_i = \sum_j C_{i-j}\, y_j$$

Implications of the Structure of Spatial Invariance

Poiseuille flow stabilization:



$$u_i = \sum_j C_{i-j}\, y_j$$

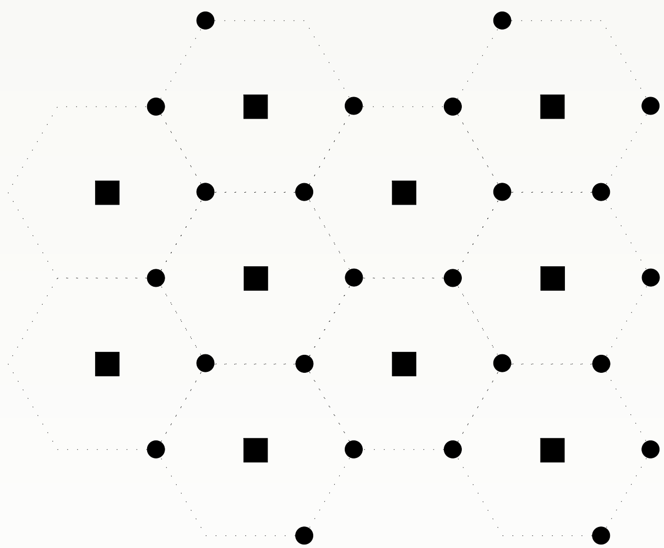
Implications of the Structure of Spatial Invariance (Cont.)

Uneven distribution of sensors and actuators

Consider the following geometry of sensors and actuators:

- ● Sensor

- ■ Actuator



What kind of spatial invariance do optimal controllers have?

Implications of the Structure of Spatial Invariance (Cont.)

Uneven distribution of sensors and actuators (Cont.)

Consider the following geometry of sensors and actuators:

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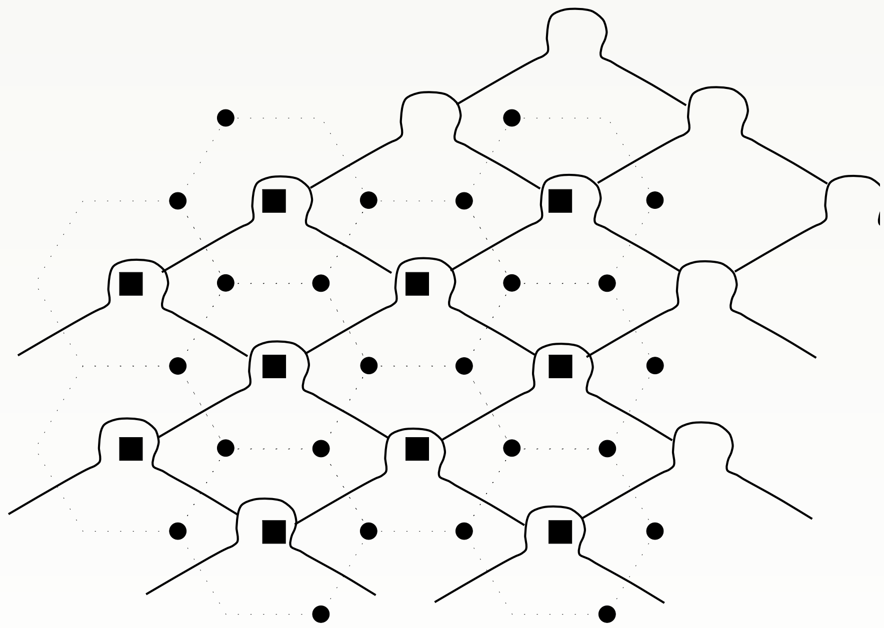


Each "cell" is a 1-input, 2-output system. **underlying group is** $\mathbb{Z} \times \mathbb{Z}$

Transform Methods

Consider the following PDE with distributed control:

$$
\begin{aligned}
\frac{\partial \psi}{\partial t}(x_1,\ldots,x_n,t) &= \mathcal{A}\left(\tfrac{\partial}{\partial x_1},\ldots,\tfrac{\partial}{\partial x_n}\right)\psi(x_1,\ldots,x_n,t) + \mathcal{B}\left(\tfrac{\partial}{\partial x_1},\ldots,\tfrac{\partial}{\partial x_n}\right)u(x_1,\ldots,x_n,t) \\
y(x_1,\ldots,x_n,t) &= \mathcal{C}\left(\tfrac{\partial}{\partial x_1},\ldots,\tfrac{\partial}{\partial x_n}\right)\psi(x_1,\ldots,x_n,t),
\end{aligned}
$$

where $\mathcal{A},\mathcal{B},\mathcal{C}$ are matrices of polynomials in $\frac{\partial}{\partial x_i}$.

Consider also combined PDE difference equations such as:

$$
\begin{aligned}
\frac{\partial \psi}{\partial t}(x_1,\ldots,x_m,k_1,\ldots,k_n,t) &= \mathcal{A}\left(\tfrac{\partial}{\partial x_1},\ldots,\tfrac{\partial}{\partial x_n},z_1^{-1},\ldots,z_n^{-1}\right)\psi(x_1,\ldots,x_n,k_1,\ldots,k_n,t) \\
&+ \mathcal{B}\left(\tfrac{\partial}{\partial x_1},\ldots,\tfrac{\partial}{\partial x_n},z_1^{-1},\ldots,z_n^{-1}\right)u(x_1,\ldots,x_n,k_1,\ldots,k_n,t)
\end{aligned}
$$

We only require that the spatial variables x, k, belong to a commutative group

Taking the Fourier transform:

$$
\hat{\psi}(\lambda,t) := \int_{\mathbb{G}} e^{-j<\lambda,x>}\psi(x,t)\,dx,
$$

The above system equations become:

$$\begin{aligned} \frac{d\hat{\psi}}{dt}(\lambda, t) &= \mathcal{A}(\lambda)\,\hat{\psi}(\lambda, t) \;+\; \mathcal{B}(\lambda)\,\hat{u}(\lambda, t) \\ \hat{y}(\lambda, t) &= \mathcal{C}(\lambda)\,\hat{\psi}(\lambda, t), \end{aligned}$$

where $\lambda \in \hat{\mathbb{G}}$, the dual group to $\mathbb{G}$.

Remark: This can be thought of as a parameterized family of finite-dimensional systems.

Block Diagonalization by Fourier Transforms

The Fourier transform converts:

spatially-invariant operators on $\mathcal{L}_2(\mathbb{G})$ $\longrightarrow$ multiplication operators on $\mathcal{L}_2(\hat{\mathbb{G}})$

In general:

group: $\mathbb{G}$	dual group: $\hat{\mathbb{G}}$	Transform
$\mathbb{R}$	$\mathbb{R}$	Fourier Transform
$\mathbb{Z}$	$\partial\mathbb{D}$	Z-Transform
$\partial\mathbb{D}$	$\mathbb{Z}$	Fourier Series
$\mathbb{Z}_n$	$\mathbb{Z}_n$	Discrete Fourier Transform

and the transforms preserve $\mathcal{L}_2$ norms:

$$\|f\|_2^2 = \int_{\mathbb{G}} |f(x)|^2 dx = \int_{\hat{\mathbb{G}}} |\hat{f}(\lambda)|^2 d\lambda = \|\hat{f}\|_2^2$$

The system operation is then spatially decoupled or “block diagonalized”:

$$\begin{aligned} \tfrac{\partial}{\partial t}\psi(x,t) &= A\,\psi(x,t) + B\,u(x,t) \\ y(x,t) &= C\,\psi(x,t) + D\,u(x,t) \end{aligned} \quad \longrightarrow \quad \begin{aligned} \tfrac{d}{dt}\hat{\psi}(\lambda,t) &= \hat{A}(\lambda)\hat{\psi}(\lambda,t) + \hat{B}(\lambda)\hat{u}(\lambda,t) \\ \hat{y}(\lambda,t) &= \hat{C}(\lambda)\hat{\psi}(\lambda,t) + \hat{D}(\lambda)\hat{u}(\lambda,t) \end{aligned}$$

A distributed,
spatially-invariant system

A parameterized family
of finite-dimensional systems

Transform Methods

In physical space

$$\begin{aligned} \frac{d}{dt}\psi_n &= A_n \star \psi_n \;+\; B_n \star u_n \\ y_n &= C_n \star \; \psi_n \end{aligned}$$

After spatial Fourier trans. (FT)

$$\begin{aligned} \frac{d}{dt}\hat{\psi}(\theta) &= \hat{A}(\theta)\,\hat{\psi}(\theta) \;+\; \hat{B}(\theta)\,\hat{u}(\theta) \\ \hat{y}(\theta) &= \hat{C}(\theta)\,\hat{\psi}(\theta) \end{aligned}$$

Implications

- Dynamics are decoupled by FT (*The* A, B, C *operators are "diagonalized"*)
- Quadratic forms preserved by FT $\Longrightarrow$ Quadratically optimal control problems are equivalent for FT
- Yields a parametrized family of mutually independent problems

Transfer functions

operator-valued transfer function

$$\mathcal{H}(s) \;=\; \mathcal{C}\,(sI - \mathcal{A})^{-1}\,\mathcal{B}$$

spatio-temporal transfer function

$$H(s,\theta) \;=\; \hat{C}(\theta)\left(sI - \hat{A}(\theta)\right)^{-1}\hat{B}(\theta)$$

A multi-dimensional system with temporal, but not spatial causality

Simple Example; Distributed LQR Control of Heat Equation

$$\frac{\partial}{\partial t}\psi(x,t) = c\frac{\partial^2}{\partial x^2}\psi(x,t) + u(x,t) \qquad \longrightarrow \qquad \frac{d}{dt}\hat{\psi}(\lambda,t) = -c\lambda^2\hat{\psi}(\lambda,t) + \hat{u}(\lambda,t)$$

Solve the LQR problem with $Q = qI$, $R = I$. The corresponding ARE family:

$$-2c\lambda^2\,\hat{p}(\lambda) - \hat{p}(\lambda)^2 + q = 0,$$

and the positive solution is:

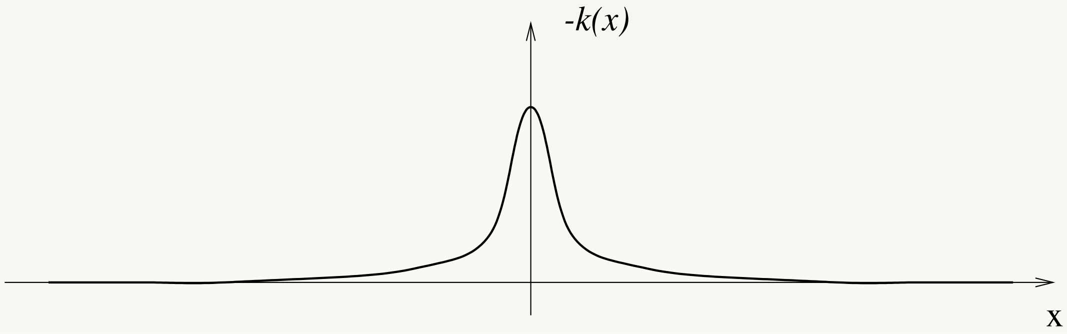
$$\hat{p}(\lambda) = -c\lambda^2 + \sqrt{c^2\lambda^4 + q}.$$

Remark: In general $\hat{P}(\lambda)$ an irrational function of λ, even if $\hat{A}(\lambda), \hat{B}(\lambda)$ are rational.
i.e. PDE systems have optimal feedbacks which are *not* PDE operators.

Let $\{k(x)\}$ be the inverse Fourier transform of the function $\{-\hat{p}(\lambda)\}$.

Then *optimal (temporally static) feedback*

$$u(x,t) \;=\; \int_{\mathbb{R}} k(x-\xi)\;\psi(\xi,t)\;d\xi$$



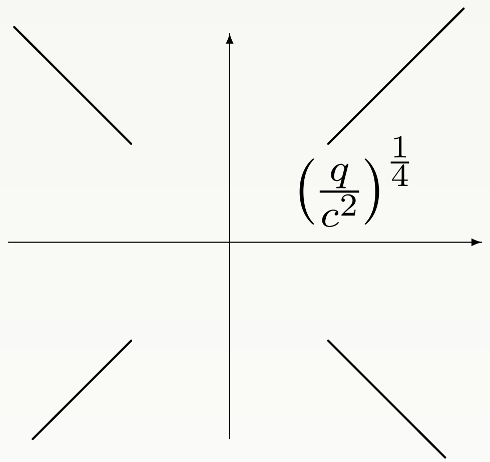
Remark: The "spread" of $\{k(x)\}$ indicates information required from distant sensors.

Distributed LQR Control of Heat Equation (Cont.)

Important Observation: $\{k(x)\}$ is "localized". It decays exponentially!!

$$\hat{k}(\lambda) = c\lambda^2 - \sqrt{c^2\lambda^4 + q}.$$

This can be analytically extended by:



$$\hat{k}_e(s) = cs^2 - \sqrt{c^2 s^4 + q},$$

which is analytic in the strip

$$\left\{ s \in \mathbb{C}\,;\; Im\{s\} < \frac{\sqrt{2}}{2}\left(\frac{q}{c^2}\right)^{\frac{1}{4}} \right\}.$$

Therefore: $\exists M$ such that

$$|k(x)| \;\leq\; Me^{-\alpha|x|}, \quad \text{for any} \;\; \alpha < \frac{\sqrt{2}}{2}\left(\frac{q}{c^2}\right)^{\frac{1}{4}}.$$

This results is true in general: under mild conditions

Solutions of AREs always inverse transform to exponentially decaying convolution kernels

Parameterized ARE solutions yield "localized" operators!

Consider unbounded domains, i.e. $\mathbb{G} = \mathbb{R}$ (or $\mathbb{Z}$).

Theorem 2. *Consider the parameterized family of Riccati equations:*

$$A^*(\lambda)P(\lambda) + P(\lambda)A(\lambda) - P(\lambda)B(\lambda)R(\lambda)B^*(\lambda)P(\lambda) + Q(\lambda) = 0, \qquad \lambda \in \hat{\mathbb{G}}.$$

Under mild conditions:
there exists an analytic continuation $P(s)$ of $P(\lambda)$ in a region

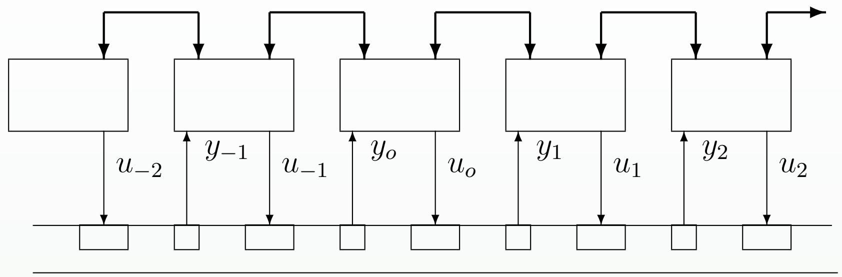
$$\{|Im(s)| < \alpha\}, \quad \alpha > 0.$$

Convolution kernel resulting from Parameterized ARE has exponential decay.
That is, they have some degree of inherent decentralization (*"localization"*)!

Comparison:

- **Modal truncation:** In the transform domain, ARE solutions decay algebraically.
- **Spatial truncation:** In the spatial domain, convolution kernel of ARE solution decays exponentially.

Therefore: Use transform domain to design $\forall \lambda$. Approximate in the spatial domain!



Observer based controller has the following structure:

Plant

$$\frac{d}{dt}\psi_n = A_n \star \psi_n + B_n \star u_n$$
$$y_n = C_n \star \psi_n$$

Controller

$$u_i = K_i \star \hat{\psi}_i$$
$$\frac{d}{dt}\hat{\psi}_n = A_n \star \hat{\psi}_n + B_n \star u_n + L_n \star (y_n - \hat{y}_n)$$

REMARKS:

- Optimal Controller is “locally” finite dimensional.
- The gains $\{K_i\}$, $\{L_i\}$ are localized (exponentially decaying) $\rightarrow$ “spatial truncation”
- After truncation, local controller need only receive information from neighboring subsystems.
- Quadratically optimal controllers are inherently distributed and semi-decentralized (*localized*)

The many remaining issues

- Various heterogeneities
  - Spatial variance
  - Irregular arrangements of sensors and actuators

- How to specify “localization” apriori

- The complexities of “high order”
  - *The phenomenology of linear infinite dimensional systems can be arbitrarily complex*

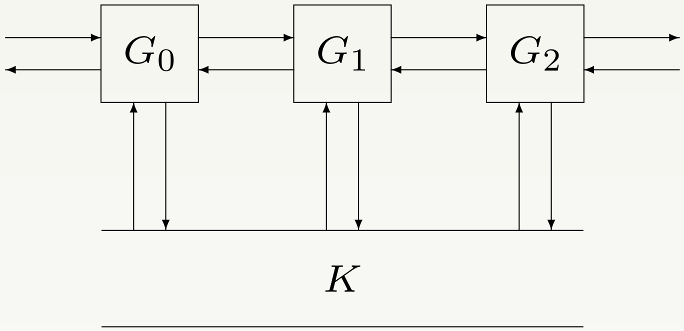
Outline

- Background
  - Distributed control and sensing
  - Useful idealizations, e.g. spatial invariance
- Structured problems
  - Constrained information passing structures
    Decentralized, Localized, etc..
  - Information passing structures which lead to convex problems
- Issues of large scale
  - Performance as a function of system size
  - Ex: Fundamental limitations in controlling *Vehicular Platoons*

Controller Architecture

Centralized vs. Decentralized control: An old and difficult problem

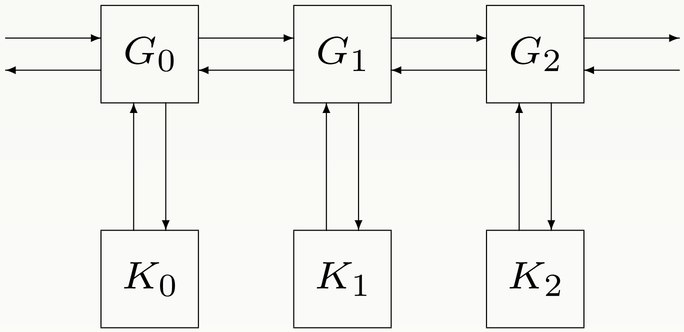
CENTRALIZED:



BEST PERFORMANCE

EXCESSIVE COMMUNICATION

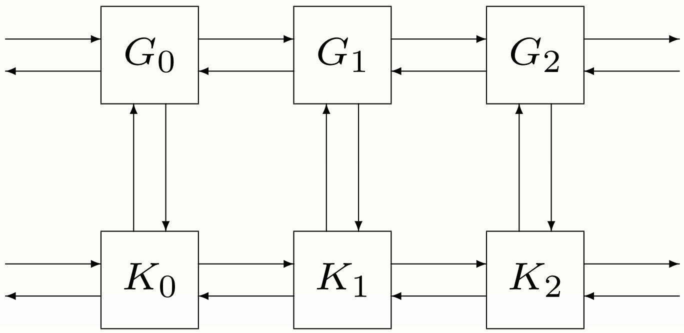
FULLY DECENTRALIZED:



WORST PERFORMANCE

NO COMMUNICATION

LOCALIZED:



MANY POSSIBLE ARCHITECTURES

System Representations

All signals are spatio-temporal, e.g. $u(x,t),\ \psi(x,t),\ y(x,t)$, etc.

Spatially distributed inputs, states, and outputs

- State space description

$$\begin{aligned} \tfrac{d}{dt}\psi(x,t) &= \mathcal{A}\,\psi(x,t) + \mathcal{B}\,u(x,t) \\ y(x,t) &= \mathcal{C}\,\psi(x,t) + \mathcal{D}\,u(x,t) \end{aligned}$$

$\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ translation invariant operators

$\longrightarrow$ spatially invariant system

- Spatio-temporal impulse response $h(x,t)$

$$y(x,t) = \int\int h(x-\xi,t-\tau)\,u(\xi,\tau)\,d\tau\,d\xi,$$

- Transfer function description

$$Y(\kappa,\omega) = H(\kappa,\omega)\,U(\kappa,\omega)$$

Spatio-temporal Impulse Response

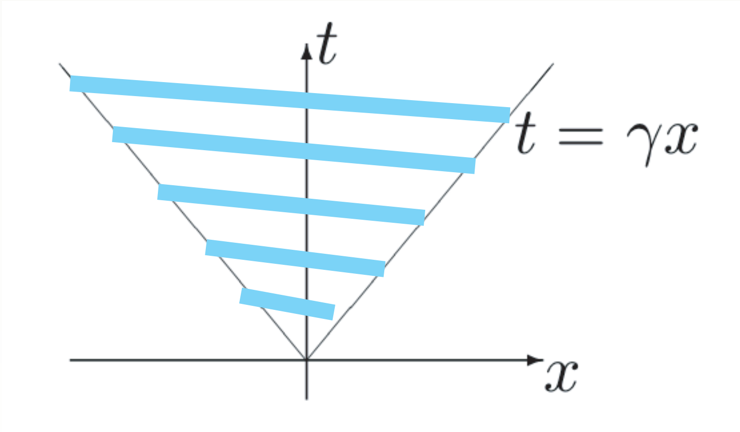
Spatio-temporal impulse response $h(x, t)$

$$y(x, t) \ = \ \int \int h(x - \xi, t - \tau) \, u(\xi, \tau) \, d\tau \, d\xi,$$

Interpretation

$h(x, t)$: effect of input on output a distance x away and time t later

Example: Constant maximum speed of effects



Def: A system is *funnel-causal* if impulse response $h(.,.)$ satisfies

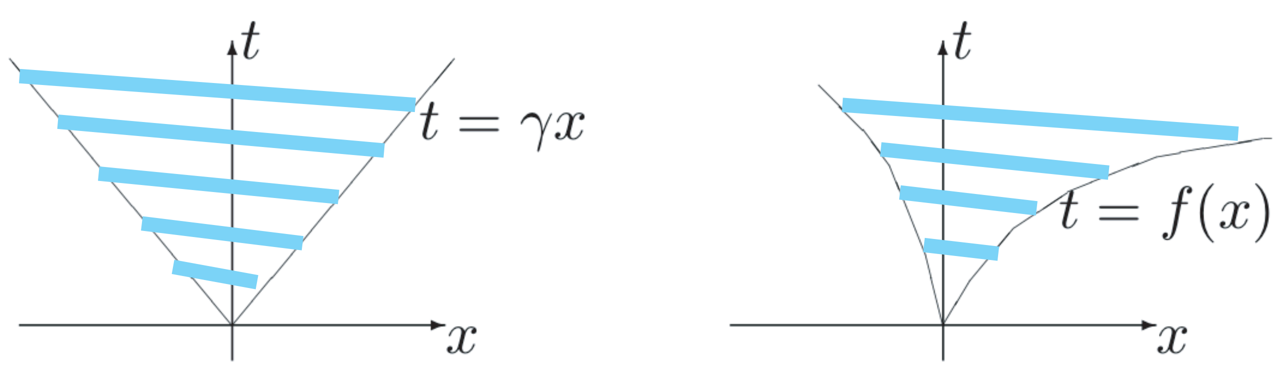
$h(x, t) = 0$ for $t < f(x)$,

where

$f(.)$ is (1) non-negative

(2) $f(0) = 0$

(3) $\{f(x),\ x \geq 0\}$ and $\{f(x),\ x \leq 0\}$ are concave



(a) Cone causality (b) Funnel causality

i.e. supp (h) is a "funnel shaped" region

Funnel Causality (Cont.)

Properties of funnel causal systems

Let S_f be a funnel shaped set

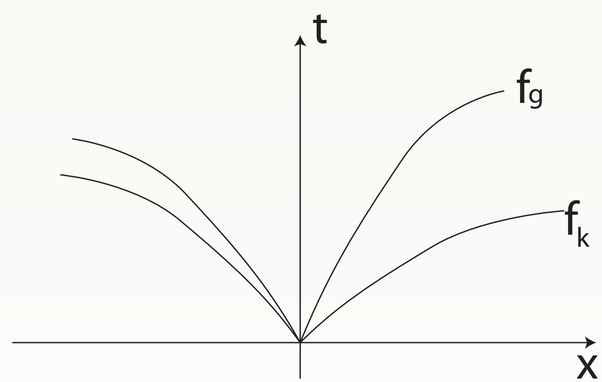
- supp $(h_1) \subset S_f$ & supp $(h_2) \subset S_f \quad \Rightarrow \quad$ supp $(h_1 + h_2) \subset S_f$
- supp $(h_1) \subset S_f$ & supp $(h_2) \subset S_f \quad \Rightarrow \quad$ supp $(h_1 * h_2) \subset S_f$
- $(I+h_1)^{-1}$ exists & supp $(h_1) \subset S_f \quad \Rightarrow \quad$ supp $\left((I+h_1)^{-1}\right) \subset S_f$

i.e.

The class of funnel-causal systems is closed under *Parallel, Serial, & Feedback interconnections*

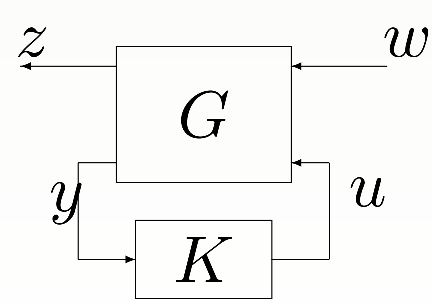
A Class of Convex Problems

- Given a plant G with supp $(G_{22}) \subset S_{f_g}$
- Let S_{f_k} be a set such that $S_{f_g} \subset S_{f_k}$
  i.e. controller signals travel at least as fast as the plant's



Solve

$$\inf_{\substack{K \text{ stabilizing} \\ \text{supp}(K) \subset S_{f_k}}} \quad \|\mathcal{F}(G;K)\|,$$



YJBK Parameterization and the Model Matching Problem

$L_f :=$ class of linear systems w/ impulse response supported in S_f

- Let $G_{22} \in L_{f_g}$
  $G_{22} = NM^{-1}$ and $XM - YN = I$ with $N, M, X, Y \in L_{f_g}$ and stable
- Let $S_{f_g} \subset S_{f_k}$
- Then all stabilizing controllers K such that $K \in L_{f_k}$ are given by

$$K \;=\; (Y + MQ)(X + NQ)^{-1},$$

  where Q is a stable system in L_{f_k}.
- The problem becomes

$$\inf_{\substack{Q \text{ stable} \\ Q \in L_{f_k}}} \quad \|H - UQV\|,$$

A convex problem!

Coprime Factorizations

Bezout identity: Find K and L such that $A+LC$ and $A+BK$ stable

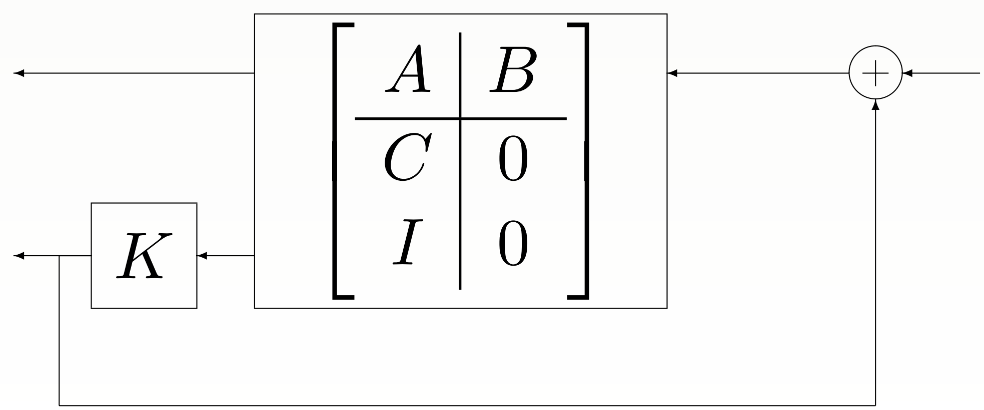
$$\begin{bmatrix} X & -Y \end{bmatrix} := \left[\begin{array}{c|cc} A+LC & -B & L \\ \hline K & I & 0 \end{array}\right], \quad \begin{bmatrix} M \\ N \end{bmatrix} := \left[\begin{array}{c|c} A+BK & B \\ \hline K & I \\ C & 0 \end{array}\right],$$

then $G = NM^{-1}$ and $XM - YN = I$,

If
- $e^{tA}B$, Ce^{tA} and $Ce^{tA}B$ are funnel causal
- K and L are funnel causal (Easy!)

then all elements of Bezout identity are funnel-causal

$$\left[\begin{array}{c|c} A+BK & B \\ \hline C & 0 \\ K & 0 \end{array}\right]$$



Example: Wave Equations with Input

1-d wave equation, $x \in \mathbb{R}$: $\partial_t^2 \, \psi(x,t) \; = \; c^2 \, \partial_x^2 \, \psi(x,t) \; + \; u(x,t)$

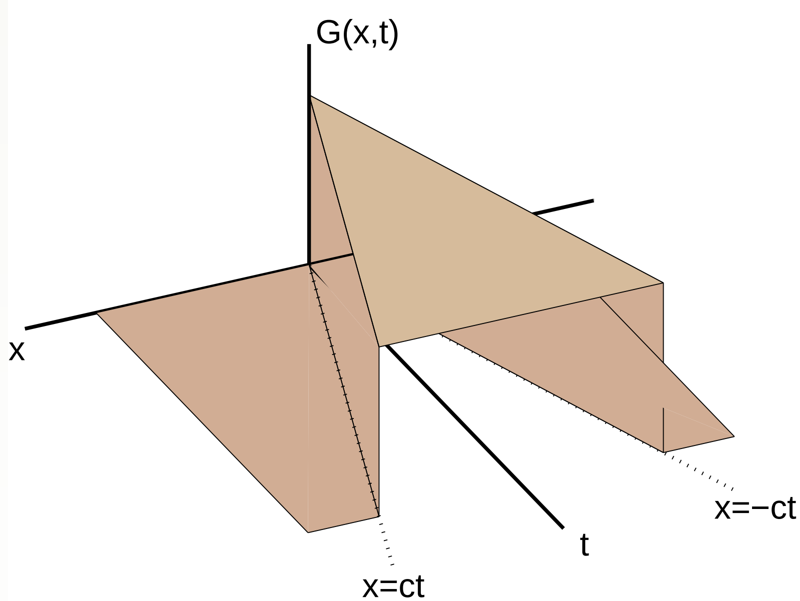
State space representation :

$$\partial_t \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} 0 & I \\ c^2 \partial_x^2 & 0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} u$$

$$\psi = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}.$$

The semigroup

$$e^{tA} = \frac{1}{2} \begin{bmatrix} T_{ct} + T_{-ct} & \frac{1}{c} R_{ct} \\ c \partial_x^2 \, R_{ct} & T_{ct} + T_{-ct} \end{bmatrix}.$$



$R_{ct} \; :=$ spatial convolution with rec$(\frac{1}{ct}x)$

$T_{ct} \; :=$ translation by ct

all components are funnel causal

e.g. the impulse response $h(x,t) \; = \; \frac{1}{2c}$ rec $\left(\frac{1}{ct}x\right)$.

Example: Wave Equations with Input (cont.)

κ := spatial Fourier transform variable ("wave number")

$$A + BK = \begin{bmatrix} 0 & 1 \\ -c^2\kappa^2 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -c^2\kappa^2 + k_1 & k_2 \end{bmatrix}.$$

Set $k_1 = 0$, then

$$\sigma(A+BK) = \bigcup_{\kappa \in \mathbb{R}} \left(k_2 \pm \frac{1}{2}\sqrt{k_2^2 - 4c^2\kappa^2} \right) = \left[\frac{3}{2}k_2, \frac{1}{2}k_2 \right] \bigcup (k_2 + j\mathbb{R})$$

Similarly for $A + LC$. Therefore, choose e.g.

$$K = \begin{bmatrix} 0 & -1 \end{bmatrix}, \quad L = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

Elements of the Bezout Identity are thus:

$$\begin{bmatrix} X & -Y \end{bmatrix} = \left[\begin{array}{cc|cc} -1 & 1 & 0 & -1 \\ -c^2\kappa^2 & 0 & -1 & 0 \\ \hline 0 & -1 & 1 & 0 \end{array}\right],$$

$$\begin{bmatrix} M \\ N \end{bmatrix} = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -c^2\kappa^2 & -1 & 1 \\ \hline 0 & -1 & 1 \\ 1 & 0 & 0 \end{array}\right].$$

Equivalently

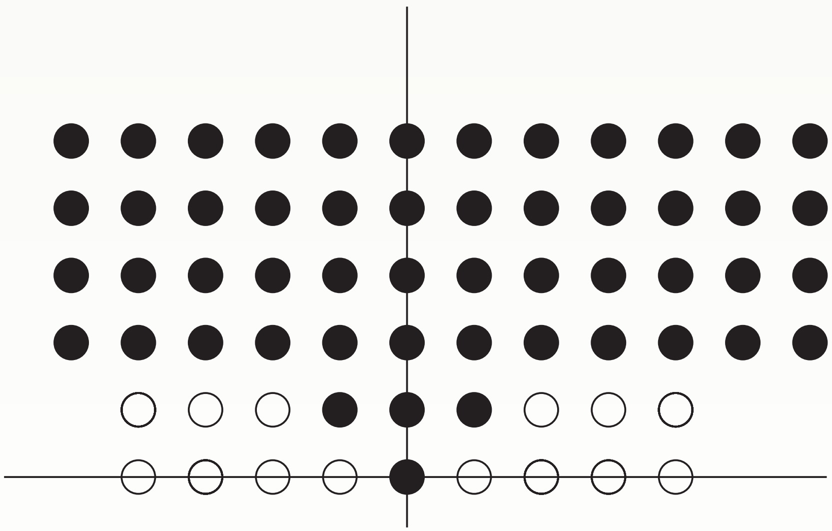
$$M = \frac{s^2 + c^2\kappa^2}{s^2 + s + c^2\kappa^2}, \qquad X = \frac{s^2 + 2s + c^2\kappa^2 + 1}{s^2 + s + c^2\kappa^2},$$

$$N = \frac{1}{s^2 + s + c^2\kappa^2}, \qquad -Y = \frac{-c^2\kappa^2}{s^2 + s + c^2\kappa^2}.$$

How easily solvable are the resulting convex problems?

- In general, these convex problems are infinite dimensional
  i.e. worse than standard half-plane causality

- In certain cases, problem similar in complexity to half-plane causality
  e.g. H^2 *with the causality structure below*
  (Voulgaris, Bianchini, Bamieh, SCL '03)



Generalizations

- Quick generalizations:
  - Several spatial dimensions
  - Spatially-varying systems

    funnel causality $\leftrightarrow$ *non-decreasing speed with distance*
  - Use relative degree in place of time delay

- Arbitrary graphs

- How to solve the resulting convex problems

Related recent work:

- *Rotkowitz & Lall*

- *Anders Rantzer*