A State-Space Approach to Control of Interconnected Systems

Part I: Spatially Invariant Systems

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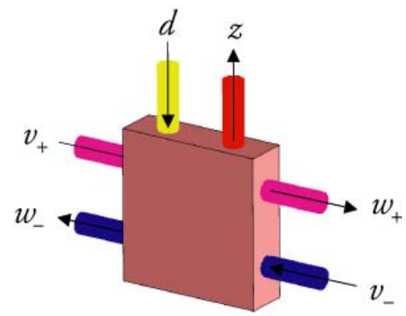
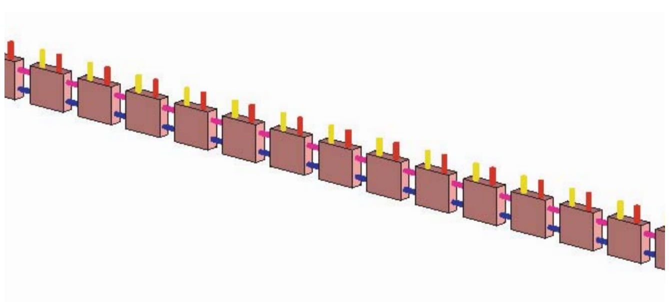
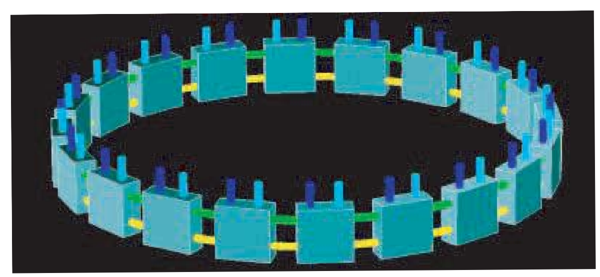
What are spatially invariant systems?

- Systems whose states depend not only on time but also a spatial variable: s belonging to a group $\mathbb{S}$.
- Invariance: Equations are invariant under $s \rightarrow s + 1$







Why are they useful?

Provide good abstractions to study

- Systems with actuation/dynamics operating on short length-scales ($\mathbb{S} = \mathbb{Z}$).

MEMS arrays...

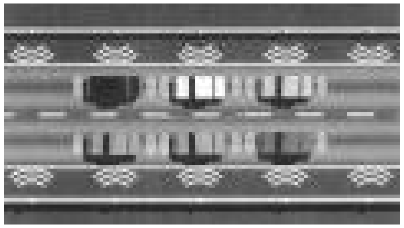
- Periodic systems ($\mathbb{S} = \mathbb{Z}_p$).

circular extrusion machines...

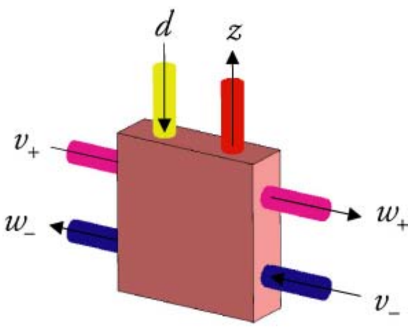
First approximation for finite length, homogeneous systems:

Deformable mirror

Automated higway



Basic Building Block



$$\begin{bmatrix} \dot{x} \\ w^{+} \\ w^{-} \\ z \end{bmatrix}(t,s) = \begin{bmatrix} A_{TT} & A_{TS_+} & A_{TS_-} & B_T \\ A_{ST_+} & A_{SS_{+,+}} & A_{SS_{+,-}} & B_{S_+} \\ A_{ST_-} & A_{SS_{-,+}} & A_{SS_{-,-}} & B_{S_-} \\ C_T & C_{S_+} & C_{S_-} & D \end{bmatrix} \begin{bmatrix} x \\ v^{+} \\ v^{-} \\ d \end{bmatrix}$$

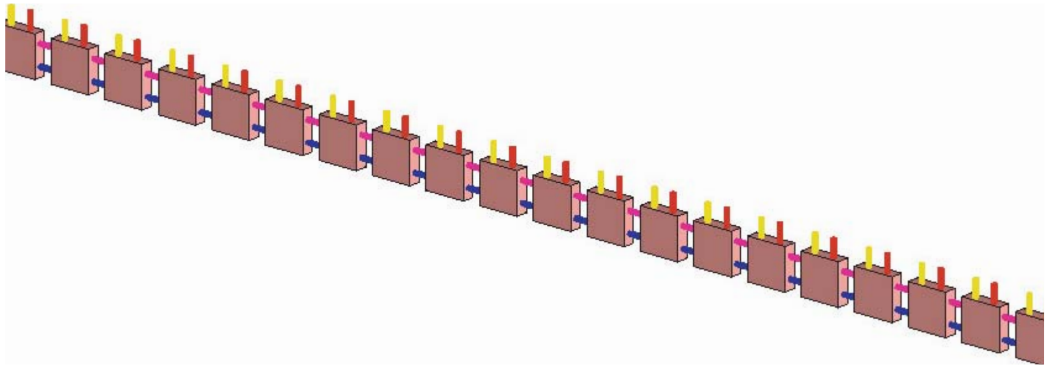
Interconnection Relation

$$v^{+}(t,s+1) = w^{+}(t,s)\ ;\ v^{-}(t,s-1) = w^{-}(t,s)$$

for all $s \in \mathbb{Z}$.

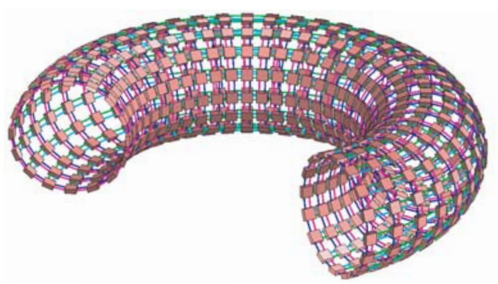
Models

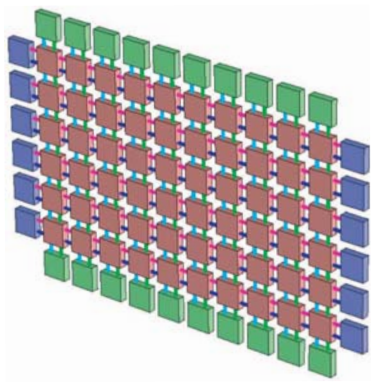
We obtain a spatially-invariant (continuous-time) system over $\mathbb{Z}$

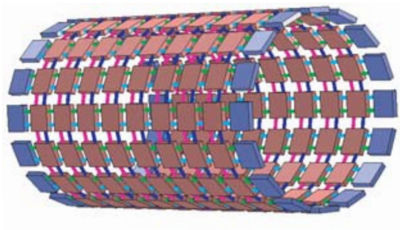


Similar to Roesser Systems

Other spatially-interconnected systems are constructed using different interconnection relations







More later...

Control goals

We want to ensure

- Well-posedness: interconnection signals $v^{\pm}$, $w^{\pm}$ have finite norms.
- Stability: $|x(t)| \leq e^{-\alpha t}|x(0)|$ for $\alpha > 0$.
- Performance: $\|z\| < \|d\|$.

where

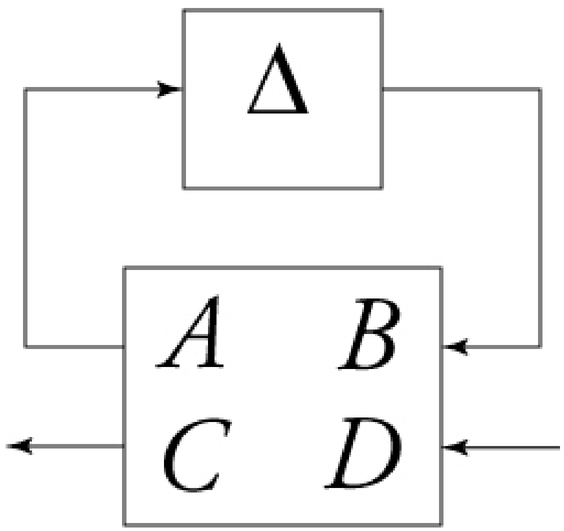
$$|x(t)| = \sum_{s=-\infty}^{\infty} x(t,s)^* x(t,s) \; ; \; \|z\| = \int_0^{\infty} |z(t)| dt$$

Treat spatially-invariant systems as interconnection in the Robust Control/ LFT framework.

Inspiration

Proving discrete-time KYP via μ-analysis methods

A is Schur stable and $\bar{\sigma}(D + C(I - zA)^{-1}zB) < 1$ for all $|z| = 1$ if and only if the following interconnection is well-connected



i.e. $\left(I - \Delta \begin{bmatrix} A & B \\ C & D \end{bmatrix}\right)$ is invertible for all $\Delta \in \mathbb{D}$.

$$\mathbb{D} = \{\Delta = \delta I \mid \delta \in \mathbb{C}, \ |\delta| = 1\}$$

- This is equivalent to

$$\mu_{\mathbb{D}}\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) < 1. \tag{1}$$

- Note that $\mathbb{D}$ is a μ-simple structure, i.e. there is equality in the inequality

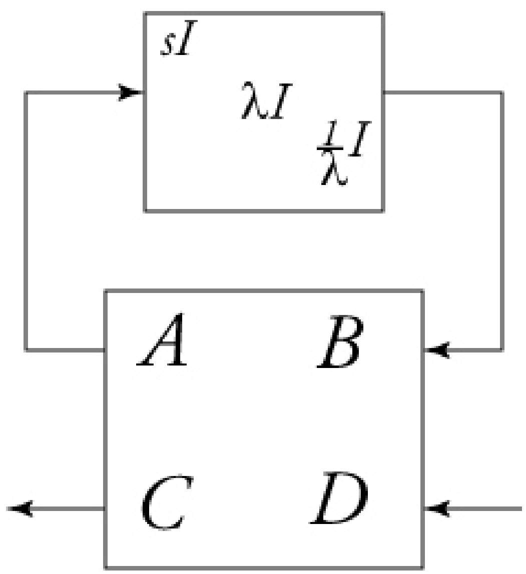
$$\mu_{\mathbb{D}}\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leq \inf_{\mathcal{X} \in Com^{+}(\mathbb{D})} \bar{\sigma}\left(\mathcal{X}^{\frac{1}{2}} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \mathcal{X}^{-\frac{1}{2}}\right).$$

- Hence, using the structure of the commutant, (1) is equivalent to

$$\exists X > 0, \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{*} \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} - \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} < 0.$$

Extension to spatially invariant systems

From the Fourier-type results of B. Bamieh's lecture, stability and well-posedness of a discrete-time spatially-invariant system on $\mathbb{Z}$ is equivalent to the following interconnection being well-connected



An other special μ-analysis problem.

Stability of spatially invariant systems

- Question reduces to:
  "When is

$$\left(I - \begin{bmatrix} sI & 0 & 0 \\ 0 & \lambda I & 0 \\ 0 & 0 & \lambda^{-1} I \end{bmatrix} \begin{bmatrix} A_{TT} & A_{TS_+} & A_{TS_-} \\ A_{ST_+} & A_{SS_{+,+}} & A_{SS_{+,-}} \\ A_{ST_-} & A_{SS_{-,+}} & A_{SS_{-,-}} \end{bmatrix} \right)$$

  invertible for all $|s| < 1$, $|\lambda| = 1$? "

- Ultimately:
  "When is $\left(I - \begin{bmatrix} \lambda I & 0 \\ 0 & \frac{1}{\lambda} I \end{bmatrix} \begin{bmatrix} A_{SS_{+,+}} & A_{SS_{+,-}} \\ A_{SS_{-,+}} & A_{SS_{-,-}} \end{bmatrix} \right)$ invertible
  for all $|\lambda| = 1$?"

Stability of spatially invariant systems

A lemma

Previous matrix is invertible for all $|\lambda| = 1$ if there exists a symmetric matrix X_s such that

$$\begin{bmatrix} A_{\mathrm{ss}_{+,+}} & A_{\mathrm{ss}_{+,-}} \\ 0 & I \end{bmatrix}^* X_\mathrm{s} \begin{bmatrix} A_{\mathrm{ss}_{+,+}} & A_{\mathrm{ss}_{+,-}} \\ 0 & I \end{bmatrix} - \begin{bmatrix} I & 0 \\ A_{\mathrm{ss}_{-,+}} & A_{\mathrm{ss}_{-,-}} \end{bmatrix}^* X_\mathrm{s} \begin{bmatrix} I & 0 \\ A_{\mathrm{ss}_{-,+}} & A_{\mathrm{ss}_{-,-}} \end{bmatrix} < 0$$

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- X_s plays the role of an element of *Com*. It is *not* sign-definite! (no causality)
- conditions are only sufficient here, as opposed to KYP.

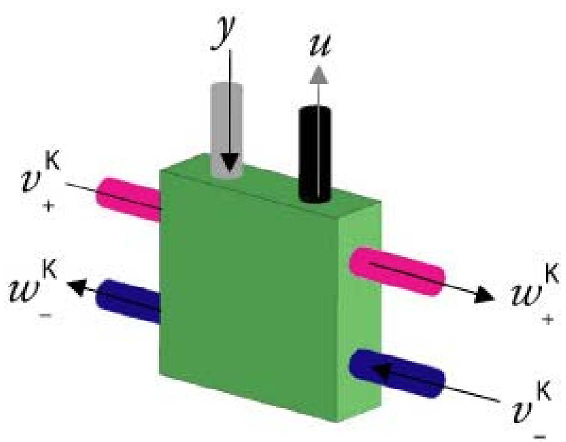
Theorem

A continuous-time spatially-invariant system on $\mathbb{Z}$ is stable if there exist a symmetric matrix X_{S} and $X_{\mathrm{T}} > 0$ such that

$$\begin{bmatrix} I & 0 & 0 \\ & A_{\mathrm{SS}_{-,-}} & B_{\mathrm{S}_{-}} \\ 0 & 0 & I \end{bmatrix}^{*} \begin{bmatrix} A_{\mathrm{TT}}^{*} X_{\mathrm{T}} + X_{\mathrm{T}} A_{\mathrm{TT}} & X_{\mathrm{T}} A_{\mathrm{TS}_{+}} & X_{\mathrm{T}} B_{\mathrm{T}} \\ (A_{\mathrm{TS}_{+}})^{*} X_{\mathrm{T}} & -X_{\mathrm{S}} & 0 \\ B_{\mathrm{T}}^{*} X_{\mathrm{T}} & 0 & -I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ & A_{\mathrm{SS}_{-,-}} & B_{\mathrm{S}_{-}} \\ 0 & 0 & I \end{bmatrix}$$
$$+ \begin{bmatrix} I & 0 & 0 \\ & A_{\mathrm{SS}_{+,+}} & B_{\mathrm{S}_{+}} \\ C_{\mathrm{T}} & C_{\mathrm{S}} & D \end{bmatrix}^{*} \begin{bmatrix} 0 & X_{\mathrm{T}} A_{\mathrm{TS}_{-}} & 0 \\ (A_{\mathrm{TS}_{-}})^{*} X_{\mathrm{T}} & X_{\mathrm{S}} & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ & A_{\mathrm{SS}_{+,+}} & B_{\mathrm{S}_{+}} \\ C_{\mathrm{T}} & C_{\mathrm{S}} & D \end{bmatrix} < 0$$

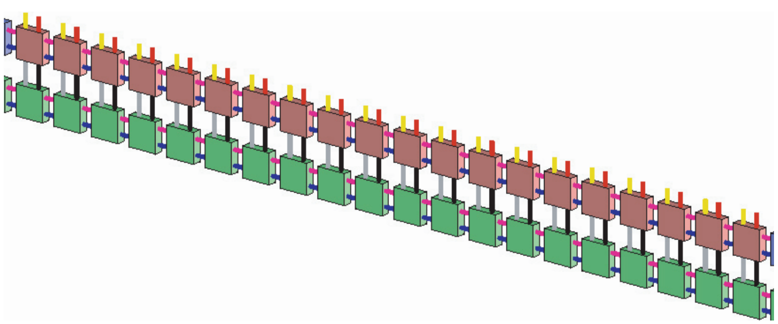
Control Synthesis

Looking for a controller with the same structure as the plant that guarantees stability and contractiveness in closed-loop.



$$\begin{bmatrix} \dot{x} \\ w \\ u \end{bmatrix} = \begin{bmatrix} A^K_{TT} & A^K_{TS} & B^K_T \\ A^K_{ST} & A^K_{SS} & B^K_S \\ C^K_T & C^K_S & D^K \end{bmatrix} \begin{bmatrix} x \\ v \\ y \end{bmatrix}$$

The closed-loop system is itself a spatially invariant system



Control Synthesis

Several steps

For the continuous-time spatially-invariant system on $\mathbb{Z}$

1. Apply analysis LMIs to the closed-loop system: obtain BMIs

Control Synthesis

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1. Apply analysis LMIs to the closed-loop system: obtain BMIs
2. Apply a Bilinear Algebraic Transformation ($\beta = \frac{1-\lambda}{1+\lambda}$) to the closed-loop system: BMIs have the same form as in the usual continuous-time synthesis problem.

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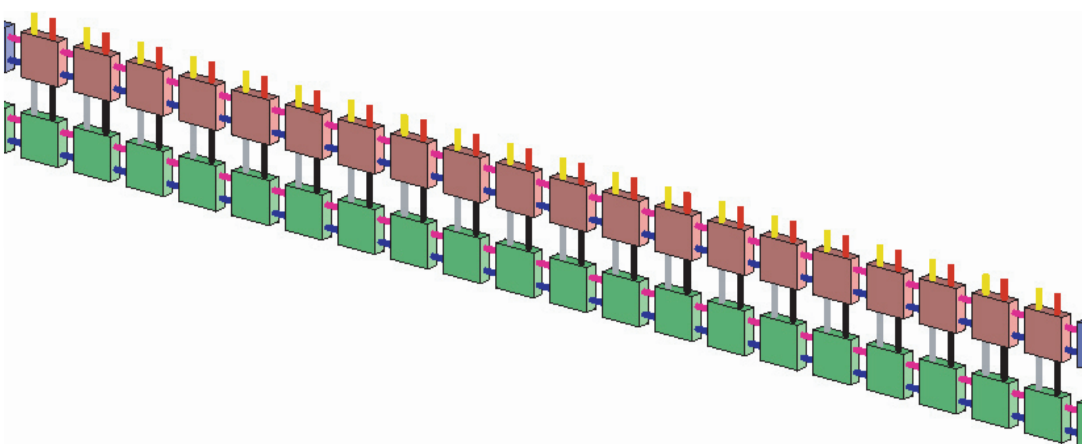
4. To reconstruct controller, apply the "inverse" of the BAT to $\bar{K}$, *making sure to obtain an implementable controller*

Summary

We have obtained convex controller synthesis conditions for spatially-invariant systems (over $\mathbb{Z}$)

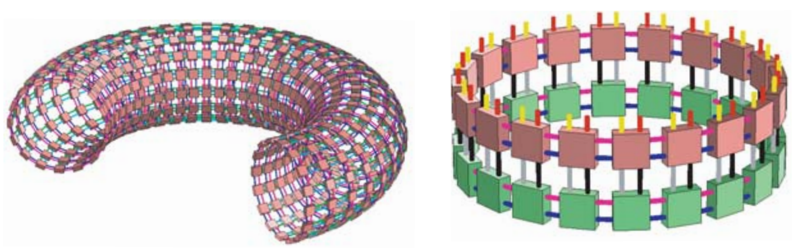
- A single LMI to solve, involving only the basic building block, even though problem is infinite-dimensional (compare the necessary and sufficient Riccati equations in B. Bamieh's talk)
- Controller has the same structure as the plant
- No conservatism added at the controller synthesis level



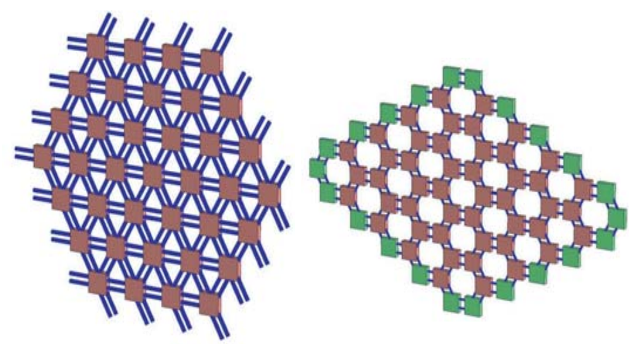
Generalizations

- Straightforward extension to system over $\mathbb{Z}^N$, $N > 1$.
- Same results hold for:

Periodic systems, independently of number of subsystems



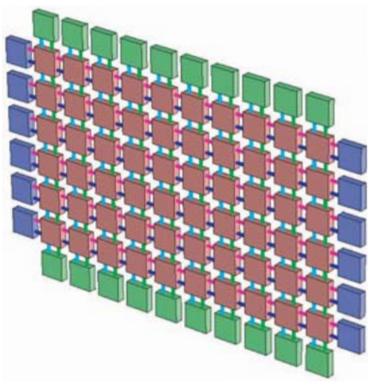
Systems over (non-commutative) Cayley graphs

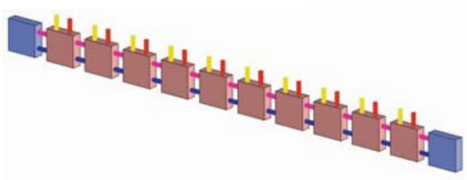


- In the latter case, can reduce conservatism by first grouping subsystems according to central subgroups.

Generalizations-II

Some boundary conditions

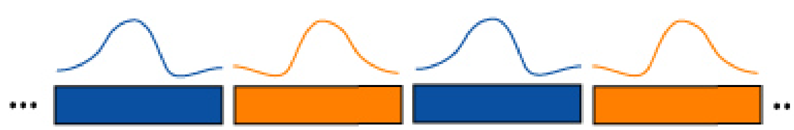




- Spatially-invariant models may be inadequate because of the effect of boundary conditions (cf. G. Stewart & G. Dumont's talk).
- Additional symmetries of the system can sometimes help account for them in a simple way.

Example/inspiration: Method of Images in PDEs

Heat Equation: $\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + Q, \ \frac{\partial T}{\partial x}(t,0) = \frac{\partial T}{\partial x}(t,1) = 0$



Reversible Systems

Definition

A finite extent system is called (spatially) reversible if there exist involution matrices P, Q, R and U such that

$$\begin{bmatrix} P & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & R \end{bmatrix} \begin{bmatrix} A_{\mathrm{TT}} & A_{\mathrm{TS}} & B_{\mathrm{T}} \\ A_{\mathrm{ST}} & A_{\mathrm{SS}} & B_{\mathrm{S}} \\ C_{\mathrm{T}} & C_{\mathrm{S}} & D \end{bmatrix} = \begin{bmatrix} A_{\mathrm{TT}} & A_{\mathrm{TS}} & B_{\mathrm{T}} \\ A_{\mathrm{ST}} & A_{\mathrm{SS}} & B_{\mathrm{S}} \\ C_{\mathrm{T}} & C_{\mathrm{S}} & D \end{bmatrix} \begin{bmatrix} P & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & U \end{bmatrix}$$

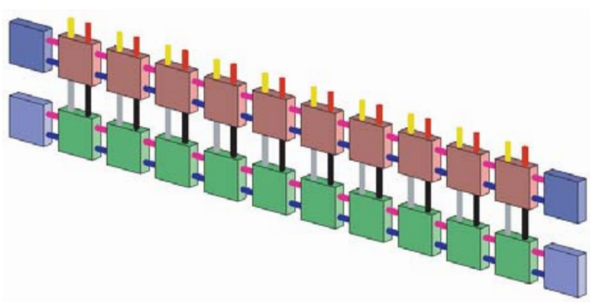
Equivariance under the action of $\mathbb{Z}_2$

A reversible system behaves as if being a part of a periodic system

Analysis

If the periodic extension is stable and contractive, then so is the corresponding reversible finite extent system.

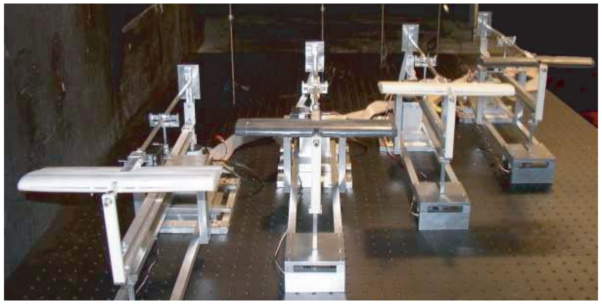
Synthesis

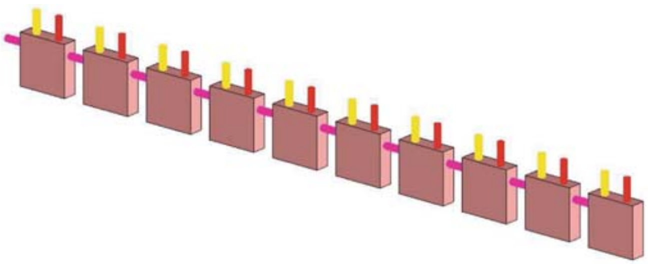


From a periodic controller such that the analysis LMIs are satisifed in closed-loop, one can construct a reversible finite-extent controller, with boundary condition matrix $(M^*)^{-1}$, such that the finite extent closed-loop is stable and contractive.

Application example

Close formation flight





- Each aircraft's wake influences its immediate follower, (hopefully) diminishing its drag.
- System can be modeled as a *chained* spatially-interconnected system, to which preceding results are applicable (approximating propagation delays)

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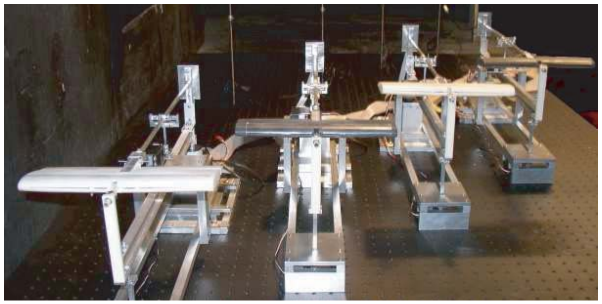

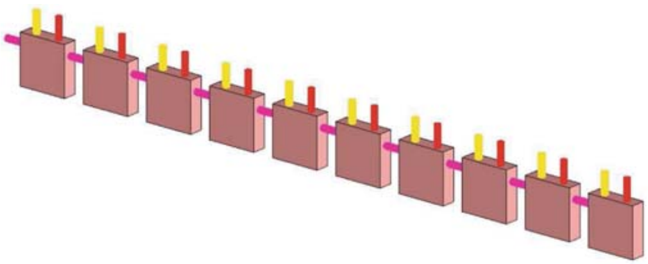

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Experimental results for a formation of 10 identical wings, applying a disturbance at each wing. $z(t, s)$ is yaw at s.

Controller	RMS Gain	Gain at rear pair
distributed	0.37	0.35
decentralized	3.15	3.13

Synthesizing the best (centralized) controller is too computationally intensive

Afternoon talk

- Heterogenous subsystems on arbitrary graphs
- Conservatism/ non-ideal interconnection relations
- Numerical Methods...

Some references I

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