A State-Space Approach to Control of Interconnected Systems

Part II: General Interconnections

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Outline

This morning

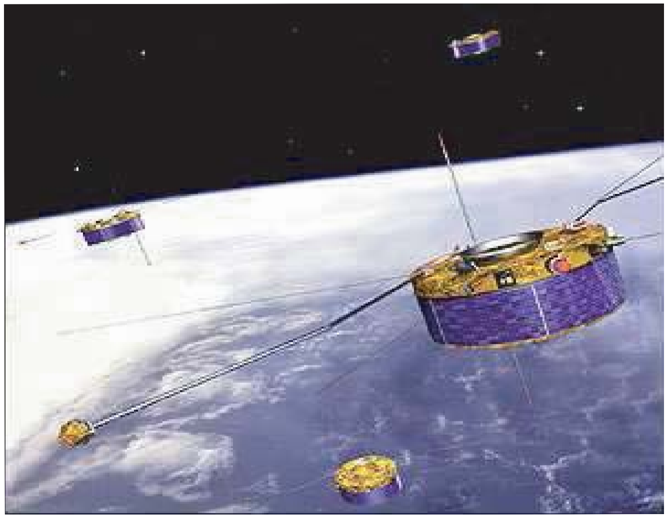
- Control of spatially-invariant systems and some extensions
- Inspiration from Robust Control methods
- Symmetries could reduce number of variables and make conditions tractable

Now

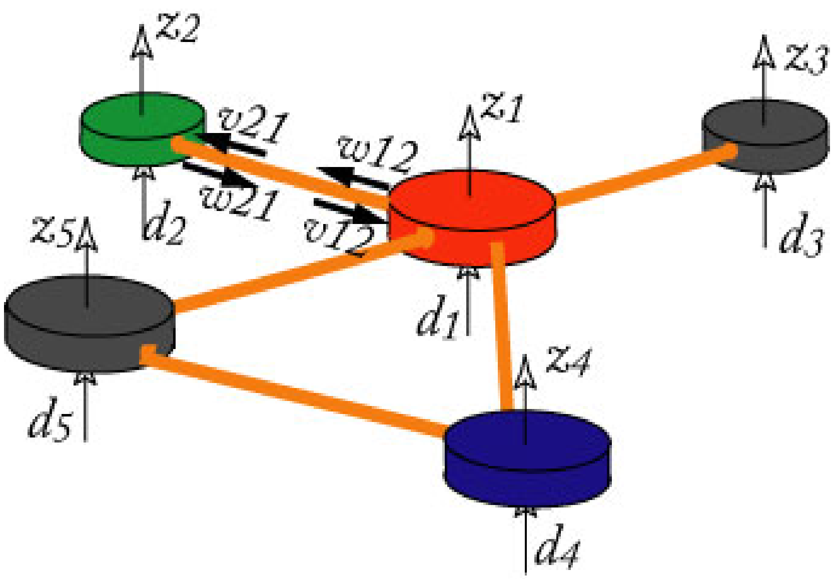
- Extend the framework to general interconnections
- Tractability and scalability remain key!

Beyond spatial invariance

There is a need for more general interconnection topologies/ relations e.g.,



Model



`[Elia, Petersen & Savkin]...`

- Each vertex is a LTI subsystem in I/O form

disturbance:	d_i
performance output:	z_i
coupling signals:	v_{ij}, w_{ij}

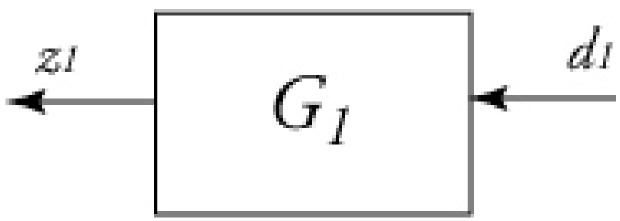
- Interconnection relation (along each edge)

$$v_{ij} = \Delta_{ij} w_{ji}$$

- Control Goals: Stability and maintaining $\sum_{i=1}^{L} \|z_i\|_2^2$ small, in spite of bounded disturbances d_i ($\mathcal{H}_\infty$–norm minimization).

Analysis

For scalability, need to relate our knowledge of the part to the properties of the whole (just like the basic building block gave information about the whole spatially invariant system).



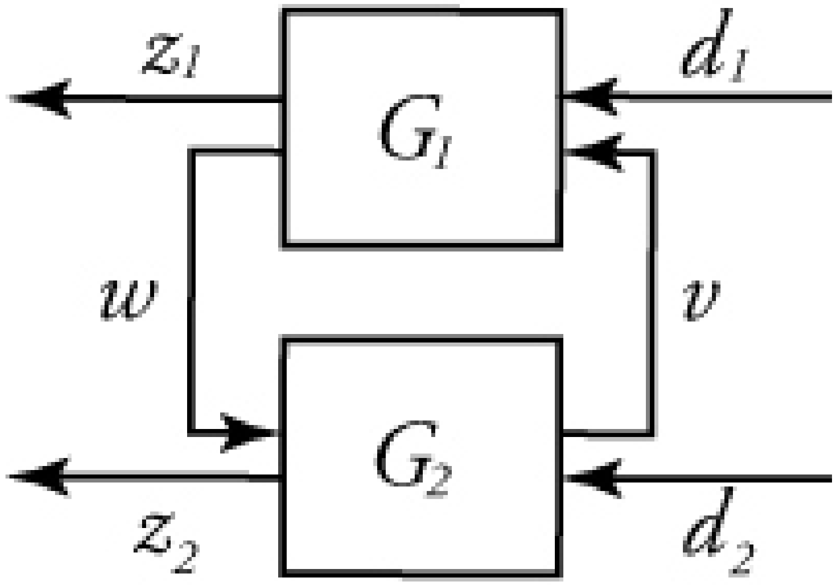
Dissipativity is such a modular notion

System G_1 is called dissipative w.r.t supply rate $s_1(d_1, z_1, t)$ if there exists a function $V_1(x_1, t)$ such that

$$V_1(x_1, t) > 0 \text{ and } \frac{dV_1(x_1(t), t)}{dt} < s_1(d_1(t), z_1(t), t).$$

V_1 is called a storage function.

Analysis –II



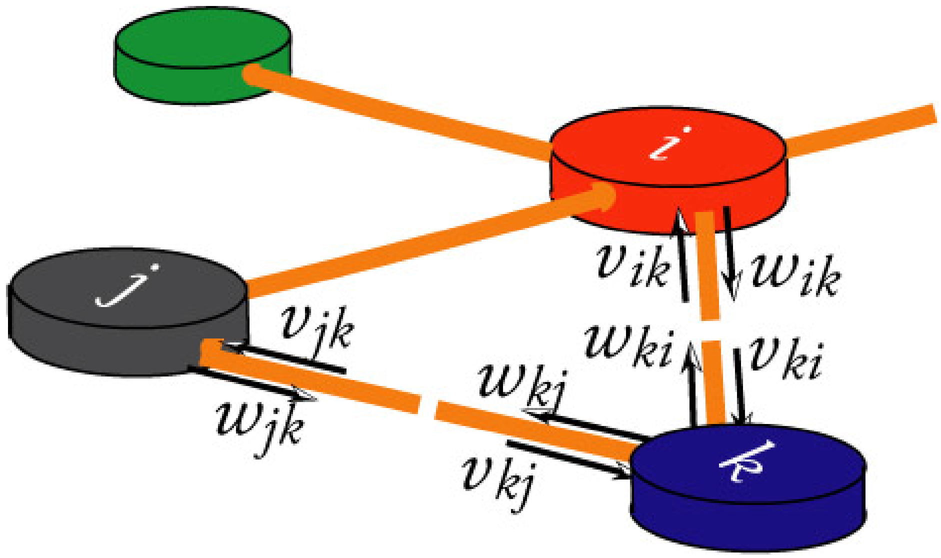
If both subsystems are dissipative with

storage function	$V_1(x_1, t)$,	supply rate	$s_1(d_1, z_1, t) + s(v, w, t)$
"	$V_2(x_2, t)$,	"	$s_2(d_2, z_2, t) + s'(w, v, t)$

and $s(v, w, t) = -s'(w, v, t)$

then interconnection is also dissipative and 'hence' stable

General Case



If i^{th} subsystem is dissipative w.r.t. supply rate

$$s_i(w_i, v_i, t) := \sum_{j \in \mathcal{N}(i)} s_{ij}(w_{ij}, v_{ij}, t) \ \forall i$$

$$\text{with } s_{ij}(w_{ij}, v_{ij}, t) = -s_{ji}(v_{ji}, w_{ji}, t) \ \forall i, j \ t \geq 0$$

then the whole system is stable.

General Case– II

For LTI subsystems with quadratic supply rates and storages, checking dissipativity amounts to solving Linear Matrix Inequalities (LMIs):

$$\mathcal{L}_i(X_T^i, \{X_{ij}\}_{j\in\mathcal{N}(i)}) \leq 0 \; ; \; X_T^i > 0 \; \forall i$$

with

$$X_{ji} = -\begin{bmatrix} 0 & I_{n_{ij}} \\ I_{n_{ij}} & 0 \end{bmatrix} X_{ij} \begin{bmatrix} 0 & I_{n_{ij}} \\ I_{n_{ij}} & 0 \end{bmatrix} .$$

Relation with literature

From the analysis viewpoint, these conditions are just particular cases of/ similar to

- General dissipativity theory `[Willems]`
- Small gain approach `[Vidyasagar, Mageirou & Ho]`
- Analysis via IQCs `[Megretski & Rantzer]`
- Full block S-procedure `[Scherer]`

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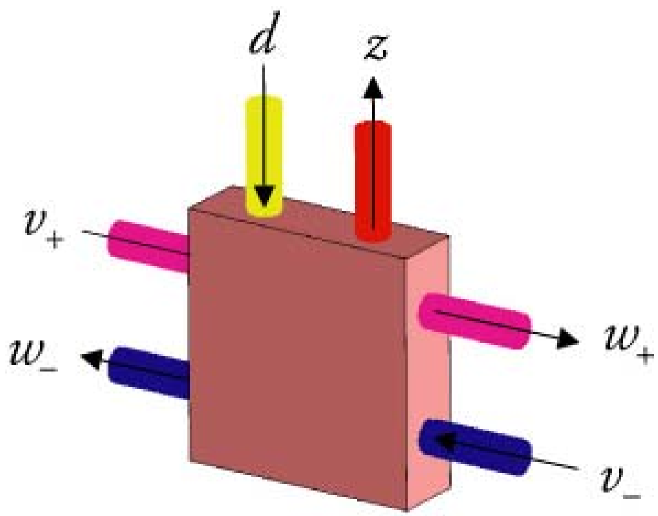
This last two viewpoints allow us to

- tackle non-ideal interconnection ($\Delta_{ij} \neq I$)
- motivate the choice of simple supply rates
- consider control synthesis

Relation with spatially invariant case



Previous LMI conditions are equivalent to subsystem being dissipative w.r.t supply rate:

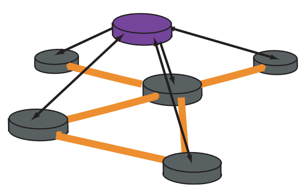
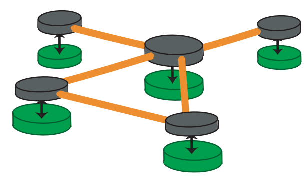
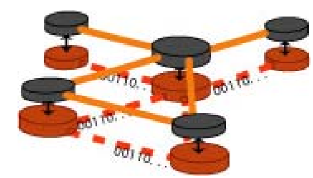
$$\begin{bmatrix} w^+ \\ v^- \end{bmatrix}^* X_S \begin{bmatrix} w^+ \\ v^- \end{bmatrix} - \begin{bmatrix} w^- \\ v^+ \end{bmatrix}^* X_S \begin{bmatrix} w^- \\ v^+ \end{bmatrix}$$

Up to relabeling, this is also what the general conditions reduce to.

Control Synthesis

As before, we are interested in designing a distributed controller, with the same topology as the plant.

Why?

	Centralized	Decentralized	Distributed
			
computing units	single	multiple	
topology	NOT respected	respected	
control design	tractable (in theory)	NON tractable (NON convex)	tractable (in practise?)

Control Synthesis – c'ed

Applying the analysis results to a closed-loop system, we obtain BMIs:

$$\begin{bmatrix} I \\ V_i^* \Theta_i U_i + R_i \end{bmatrix}^* M_i \begin{bmatrix} I \\ V_i^* \Theta_i U_i + R_i \end{bmatrix} < 0, \quad \forall i$$

where

$$\begin{cases} \mathrm{R}_i,\ U_i,\ V_i : & \text{contain plant's subsystem's data} \\ \Theta_i : & \text{contains controller's subsystem's data} \\ \mathrm{M}_i : & \text{contains storage function matrix } (\mathrm{X}_\mathrm{T}^i)_\mathrm{C} \\ & \text{and supply rates matrices } (\mathrm{X}_{ij})_\mathrm{C} \text{ in closed-loop.} \end{cases}$$

Control Synthesis – c’ed

Elimination Lemma [Scherer ’01]

Let M be a symmetric matrix with inertia

$$in(M) = (\underset{+}{m}, 0, \underset{-}{n}). \tag{1}$$

There exists a full matrix Θ such that

$$\begin{bmatrix} I_m \\ V^*\Theta U_i + R \end{bmatrix}^* M \begin{bmatrix} I_m \\ V^*\Theta U + R \end{bmatrix} < 0,$$

if and only if

$$U_\perp^* \begin{bmatrix} I \\ R \end{bmatrix}^* M \begin{bmatrix} I \\ R \end{bmatrix} U_\perp < 0$$

$$V_\perp^* \begin{bmatrix} -R^* \\ I \end{bmatrix}^* M^{-1} \begin{bmatrix} -R^* \\ I \end{bmatrix} V_\perp > 0$$

Distributed Control

When using a distributed architecture,

$$(X^i_{\mathrm{T}})_{\mathrm{C}} = \begin{bmatrix} (X^i_{\mathrm{T}})_{\mathrm{G}} & ? \\ ? & ? \end{bmatrix},$$

$$(X_{ij})_{\mathrm{C}} = \begin{bmatrix} (X_{ij})_{\mathrm{G}} & ? \\ ? & ? \end{bmatrix},$$

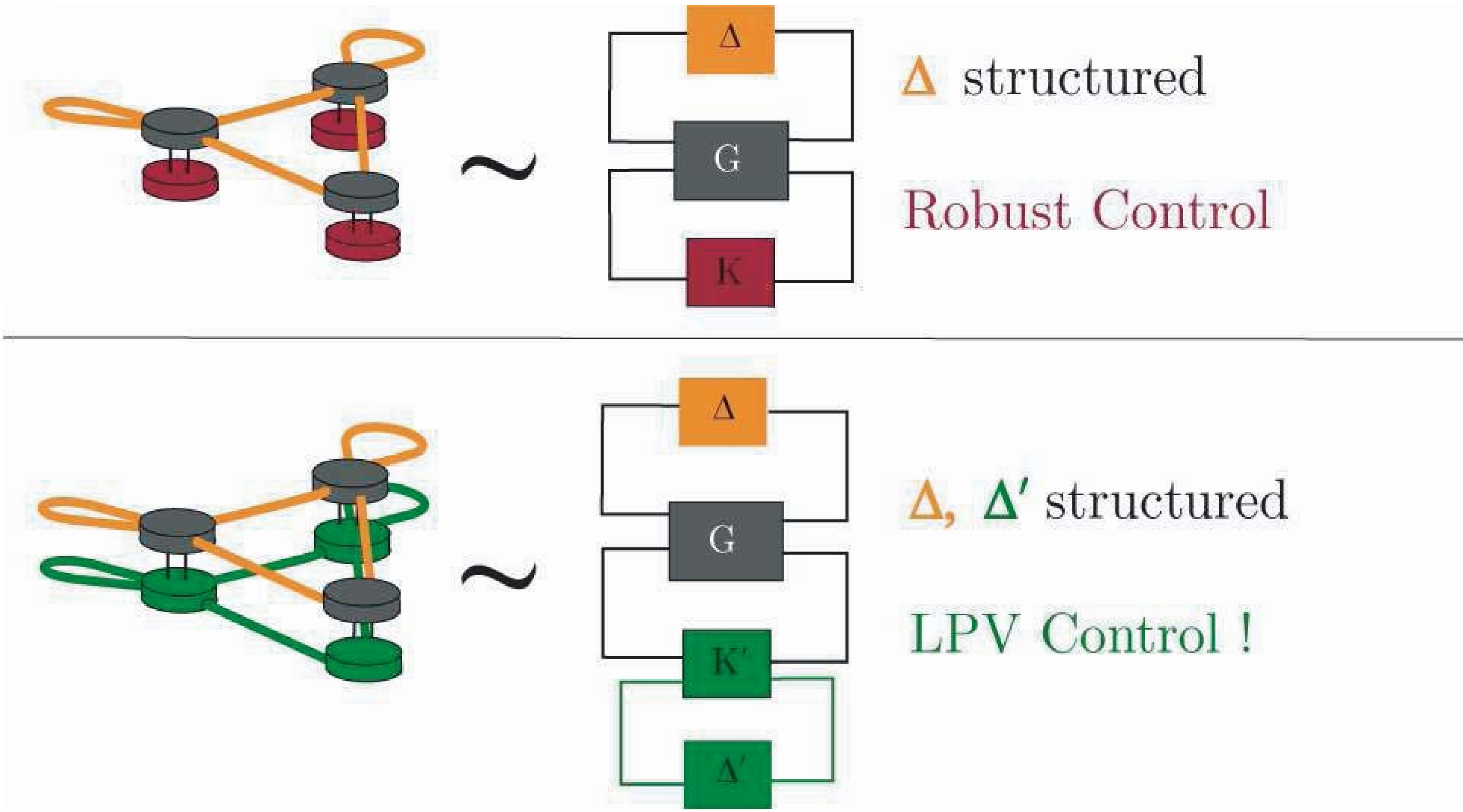
which leaves enough free variables to eliminate the non-convex constraint relating M_i and M_i^{-1}, when taking $n^{\mathrm{K}}_{ij} = 3n_{ij}$

Synthesis of dynamic output feedback distributed controllers over an arbitrary graph is a convex problem!

Not the case for decentralized controllers...

Intuitively

Why does it work?



Extensions to non-ideal interconnections

- Neutrality condition $(X_{ij})_{\mathrm{C}} = -\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} (X_{ji})^{\mathrm{C}} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ is key in enabling the use of the Elimination Lemma.
- Can convexify synthesis as soon as the IQCs describing interconnection relations
  - is equivalent to a set of constraints on each vertex

$$\text{supply rate}(w_i, v_i) = \sum_{\text{channel } (i,j)} \begin{bmatrix} w_{ij} \\ v_{ij} \end{bmatrix}^* X_{ij} \begin{bmatrix} w_{ij} \\ v_{ij} \end{bmatrix}$$

  - guarantees that (1) holds
- Note: Such interconnection relations come up naturally when studying the conservatism of LMIs...

Some examples

Contractive channels:
delays, low-pass transmission...

$$\Delta_{ij} = \delta_{ij} I \ ; \ \|\delta_{ij}\|_\infty \leq 1$$

$$X_{ij} = \begin{bmatrix} X_{ij}^{11} & 0 \\ 0 & X_{ij}^{22} \end{bmatrix} \ ; \ X_{ij}^{11} < 0, \ X_{ij}^{22} > 0$$

'Perfect erasure channels':
$\delta_{ij} = \delta$, a random variable taking value 1 with probability p, 0 otherwise

$$X_{ji} = -\begin{bmatrix} 0 & \frac{1}{\sqrt{p}} I \\ \sqrt{p} I & 0 \end{bmatrix}^* X_{ij} \begin{bmatrix} 0 & \frac{1}{\sqrt{p}} I \\ \sqrt{p} I & 0 \end{bmatrix}$$

Some examples

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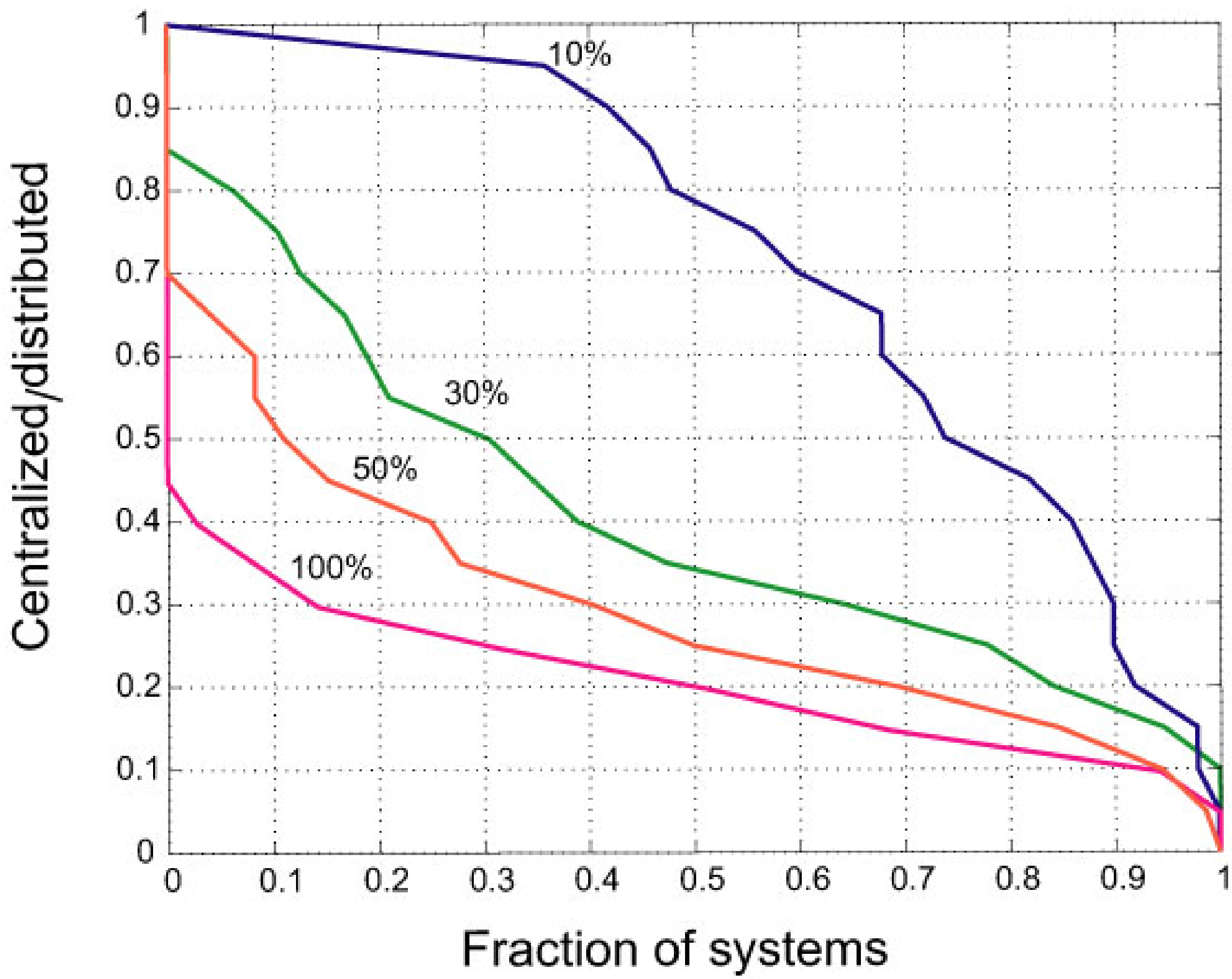
Channel with arbitrary failures:

$\delta_{ij}(t)$ switches, with values in $\{0, 1\}$.

$$X_{ij} = \begin{bmatrix} X_{ij}^{11} & X_{ij}^{12} \\ X_{ij}^{12*} & X_{ij}^{22} \end{bmatrix} \; ; \; X_{ij}^{11} < 0$$

Tractability and conservatism

“The price of decentralization”



Tractability and conservatism

Reduction

For both ideal and Markovian failing channels, *necessary and sufficient* LMI conditions for stabilization (and guaranteed $\mathcal{H}_\infty$ performance in closed-loop) exist. `[Seiler & Sengupta, '05]`. But, they

- yield centralized controllers
- involve more variables, even though theoretically tractable.

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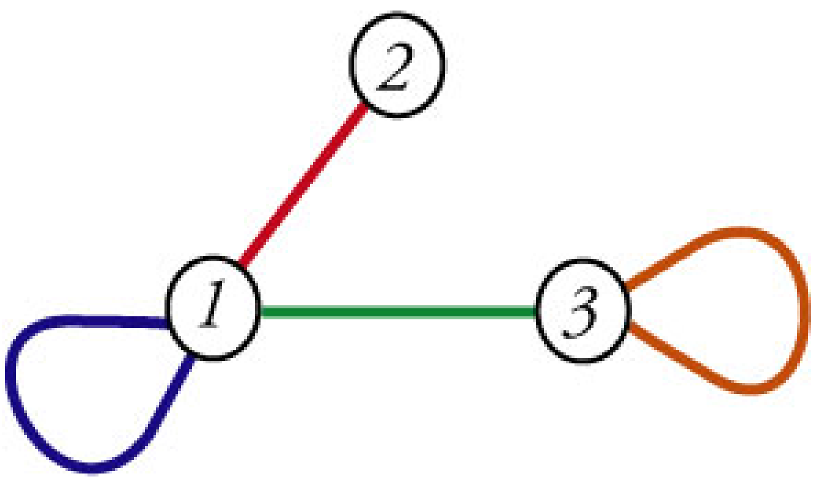
For large-scale systems with L subsystems of order $\mathcal{O}(n)$; E interconnections where signals of dimension $\mathcal{O}(s)$ are shared

	Centralized	Distributed
ideal	$\mathcal{O}(n^2 L^2)$	$\mathcal{O}(Ln^2 + Es^2)$
failing	$\mathcal{O}(n^2(L^2 + 2^E))$	"

Typically $s << n$, $E << L^2$.

Towards distributed algorithms

- At this point, we have convex tools for the control of interconnected systems.
- We have accepted some degradation in performance in favor of reduction in problem's variables' size and complexity.
- This exploitation of problem's structure can be taken further, at the algorithmic level



$$\mathcal{L}_1\left(\mathcal{X}_1,\ \mathcal{X}_{12},\ \mathcal{X}_{13}\right) < 0$$
$$\mathcal{L}_2\left(\mathcal{X}_2,\ \mathcal{X}_{12}\right) < 0$$
$$\mathcal{L}_3\left(\mathcal{X}_{13}, \mathcal{X}_3\right) < 0$$

after making constraints explicit.

Amenable to *distributed computation*

- Start by replacing analysis or synthesis LMI feasibility problem by the convex, non-smooth, minimization problem

$$\min_{\{t_i\},\{\mathcal{Y}_i\},\{\mathcal{X}_{ij}\}} \sum_i t_i$$

$$s.t.\ \mathcal{L}_i(\mathcal{Y}_i, \{\mathcal{X}_{ij}\}_{j\in\mathcal{N}_i}) \leq t_i I,\ \forall i$$

$$\mathcal{Y}_i > 0\ ;\ t_i \geq -1,\ \forall i$$

with $\mathcal{X}_{ji} = -\mathcal{X}_{ij},\ \forall i, j.$

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- By convexity, we must compute $\min_{\{\mathcal{X}_{ij}\}} \Phi := \sum_i \phi_i$ where

$$\phi_i(\{\mathcal{X}_{ij}\}_{j\in\mathcal{N}_i}) := \min_{t_i,\mathcal{Y}_i} t_i$$

$$s.t.\ \mathcal{L}_i(\mathcal{Y}_i, \{\mathcal{X}_{ij}\}_{j\in\mathcal{N}_i}) \leq t_i I$$

$$\mathcal{X}_i > 0\ ;\ t_i \geq -1$$

Decomposition and subgradient methods

- This minimization can be computed via a subgradient algorithm

$$\mathcal{X}_{ij}^{k+1} = \mathcal{X}_{ij}^{k} - \alpha_k \left(g_{ij}(\{\mathcal{X}_{ij}^{k}\}) - g_{ji}(\{\mathcal{X}_{ji}^{k}\}) \right)$$

$$\Phi_{best}^{k+1} = \min \left(\Phi_{best}^{k}, \Phi(\{\mathcal{X}_{ij}^{k+1}\}) \right)$$

where

$$\lim_{k \to \infty} \alpha_k = 0, \sum_{k=1}^{\infty} \alpha_k = \infty \ ; \ g_{ij}(\mathcal{X}_{ij}) \in \partial_j \phi_i(\mathcal{X}_{ij}) \ \forall i, j$$

- Subgradients are given by the optimal dual variable corresponding to problem ϕ_i and can be computed independently by each subsystem

Properties of the algorithm

- Communication only takes place between neighbors on the graph

  subsystems don't need to know 'who' their neighbors are (i.e., their model), only restricted information

  Delocalization through price-like variables

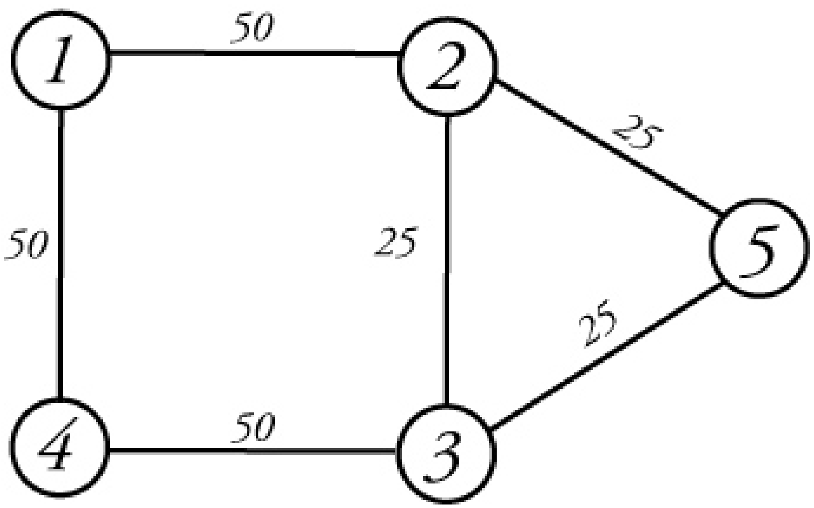
- Work in an asynchronous framework:
  subgradients can be used as they become available, as long as their are *all* incorporated with the same long term frequency [Nedić, Bertsekas & Borkar, 2001]

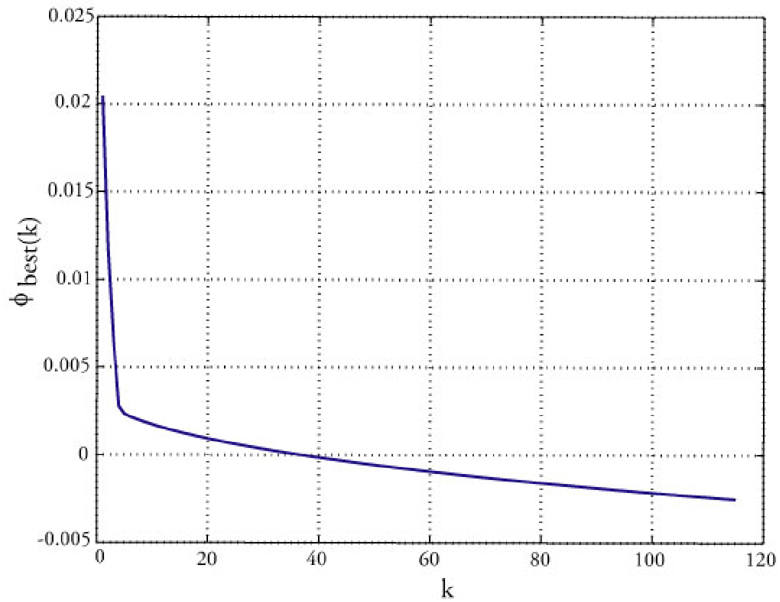
  A step towards online control design

BUT

- Convergence is slow
- Feasibility of original LMIs attained only asymptotically...

An example





- Cannot perform a single centralized Newton-step with centralized algorithm!
- Our algorithm converges in 6 minutes with a parallel implementation...

Extensions & Perspectives

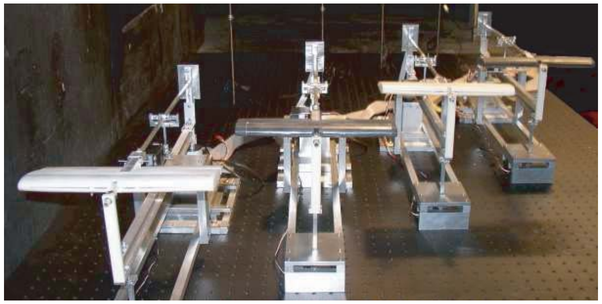
More distributed Algorithms

- Could develop other, similar algorithm e.g., using active sets.
- In this case, less subgradient computations at each step BUT
- Must determine active set in distributed fashion.
- Better suited for online design/ reconfiguration?...

`[B. Rangarajan & C.L., forthcoming]`

Extensions & Perspectives

Formation flight example revisited



- Using the same dissipativity ideas, can derive scalable stability tests for piece-wise linear interconnected systems.
- Both storage functions and supply rates are piece-wise quadratic and there is an exponential number of neutrality conditions.

- Can be used to check the previous control laws for formation flight on more accurate models.

`[Fowler & D'Andrea, forthcoming]`

Some references I

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