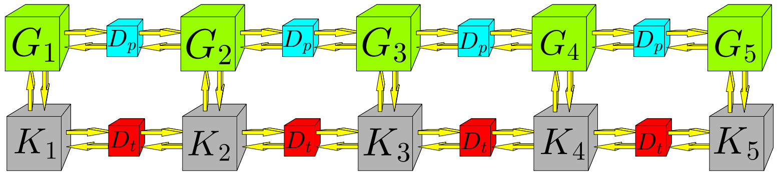
Parameterization of Stabilizing Controllers for Interconnected Systems

Michael Rotkowitz



Contributors: Sanjay Lall, Randy Cogill

Department of Signals, Sensors, and Systems
Royal Institute of Technology (KTH)

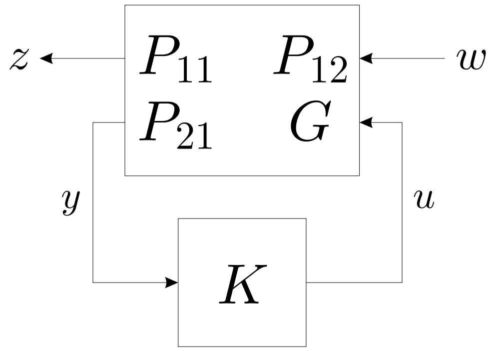
Control, Estimation, and Optimization of Interconnected Systems:
From Theory to Industrial Applications
CDC-ECC'05 Workshop, 11 December 2005

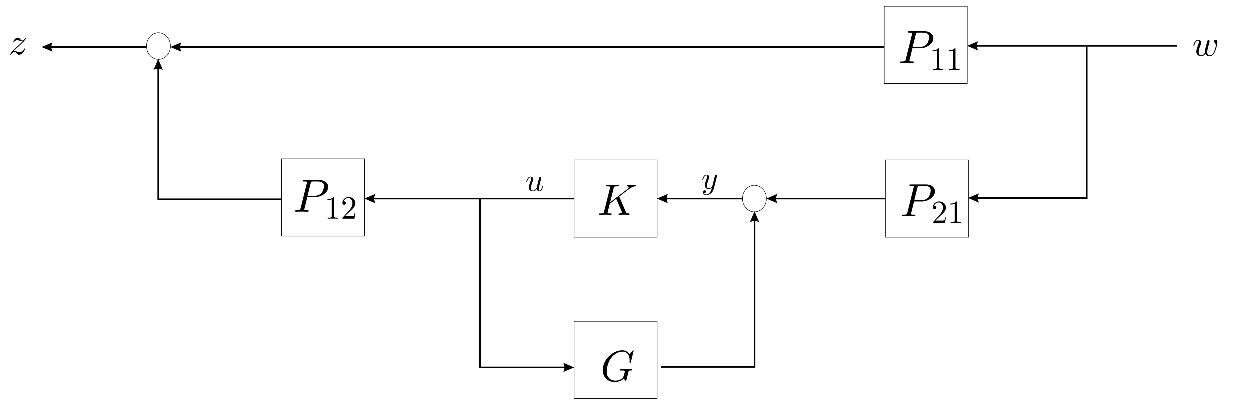
Overview

- Motivation: Optimal Constrained Control
- **Linear Time-Invariant**
  - Quadratic invariance
  - Optimal Control over Networks
  - *Summation*
- **Nonlinear Time-Varying**
  - *Iteration*
  - New condition

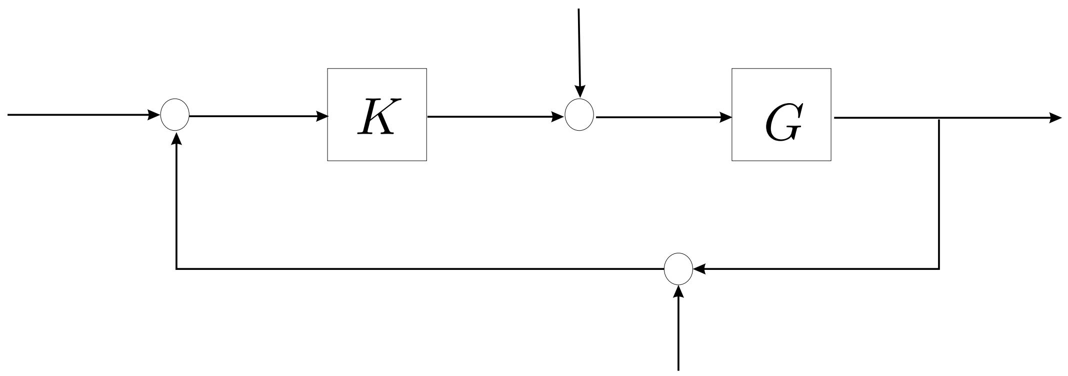
Block Diagrams

Two-Input Two-Output

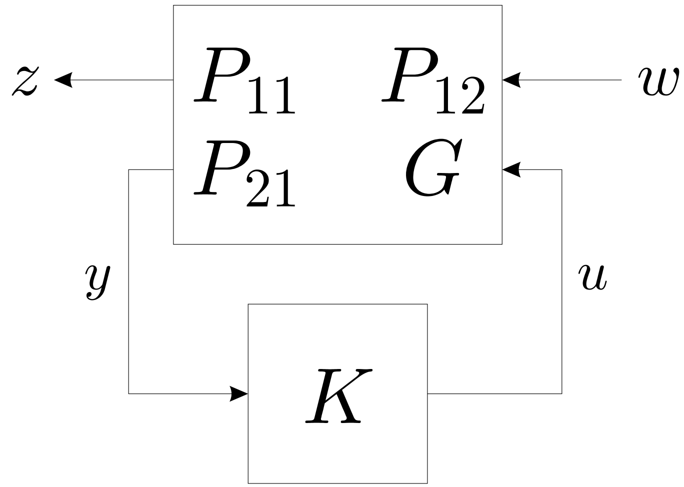




Classical

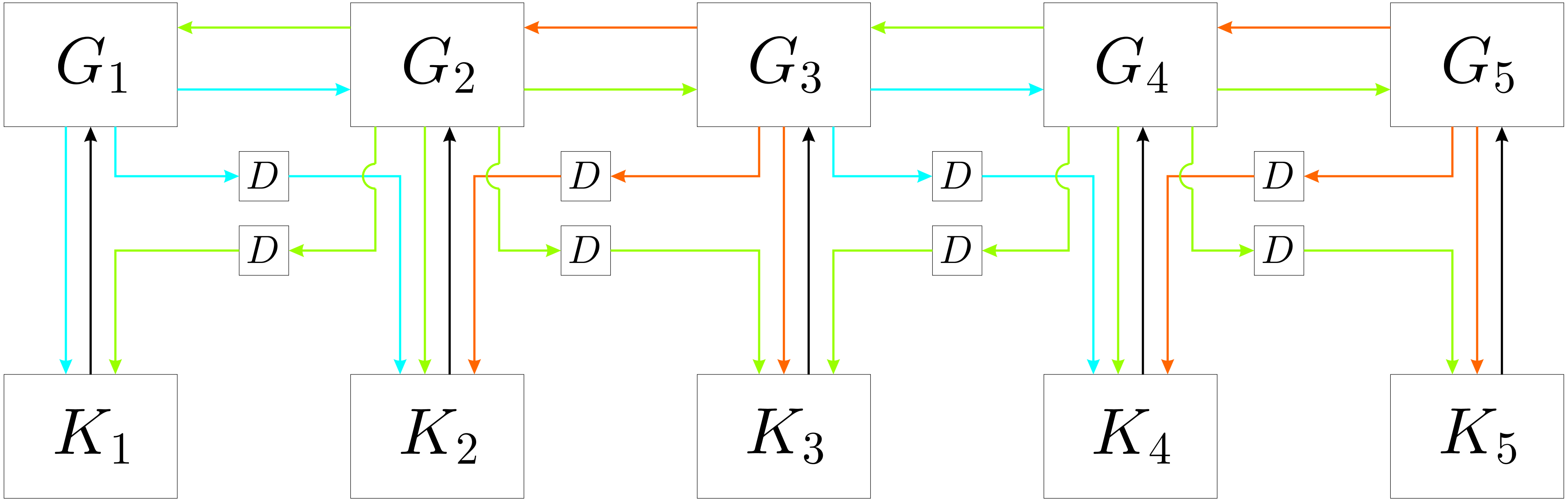


Standard Formulation



$$\begin{aligned} \text{minimize} \quad & \|P_{11} + P_{12}K(I - GK)^{-1}P_{21}\| \\ \text{subject to} \quad & K \text{ stabilizes } P \end{aligned}$$

Communicating Controllers

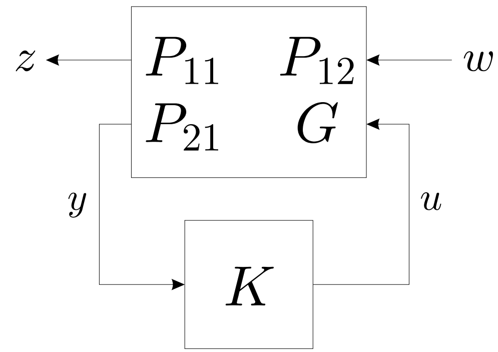


Control design problem is to find K which is block tri-diagonal.

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} K_{11} & DK_{12} & 0 & 0 & 0 \\ DK_{21} & K_{22} & DK_{23} & 0 & 0 \\ 0 & DK_{32} & K_{33} & DK_{34} & 0 \\ 0 & 0 & DK_{43} & K_{44} & DK_{45} \\ 0 & 0 & 0 & DK_{54} & K_{55} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}$$

General Formulation

The set of K with a given decentralization constraint is a subspace S, called the *information constraint*.



We would like to solve

$$\begin{aligned} \text{minimize} \quad & \|P_{11} + P_{12}K(I - GK)^{-1}P_{21}\| \\ \text{subject to} \quad & K \text{ stabilizes } P \\ & K \in S \end{aligned}$$

- For general P and S, there is no known tractable solution.

Change of Variables - Stable Plant

$$\begin{array}{rl} \text{minimize} & \|P_{11} + P_{12}K(I - GK)^{-1}P_{21}\| \\ \text{subject to} & K \text{ stabilizes } P \end{array}$$

Using the change of variables

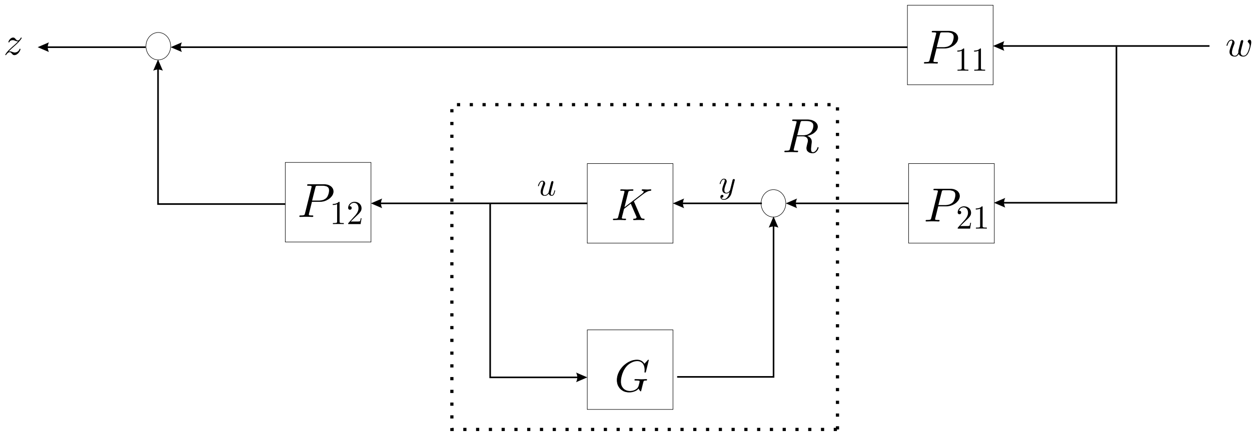
$$R = K(I - GK)^{-1}$$

we obtain the following equivalent problem

$$\begin{array}{rl} \text{minimize} & \|P_{11} + P_{12}RP_{21}\| \\ \text{subject to} & R \text{ stable} \end{array}$$

This is a *convex optimization* problem.

Change of Variables - Stable Plant



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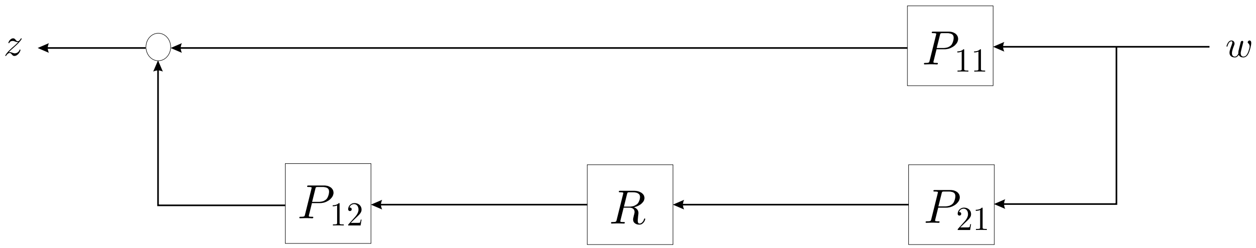

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Breakdown of Convexity - Stable Plant

$$\begin{aligned} \text{minimize} \quad & \|P_{11} + P_{12}K(I - GK)^{-1}P_{21}\| \\ \text{subject to} \quad & K \text{ stabilizes } P \\ & K \in S \end{aligned}$$

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$$\begin{aligned} \text{minimize} \quad & \|P_{11} + P_{12}RP_{21}\| \\ \text{subject to} \quad & R \text{ stable} \\ & R(I + GR)^{-1} \in S \end{aligned}$$

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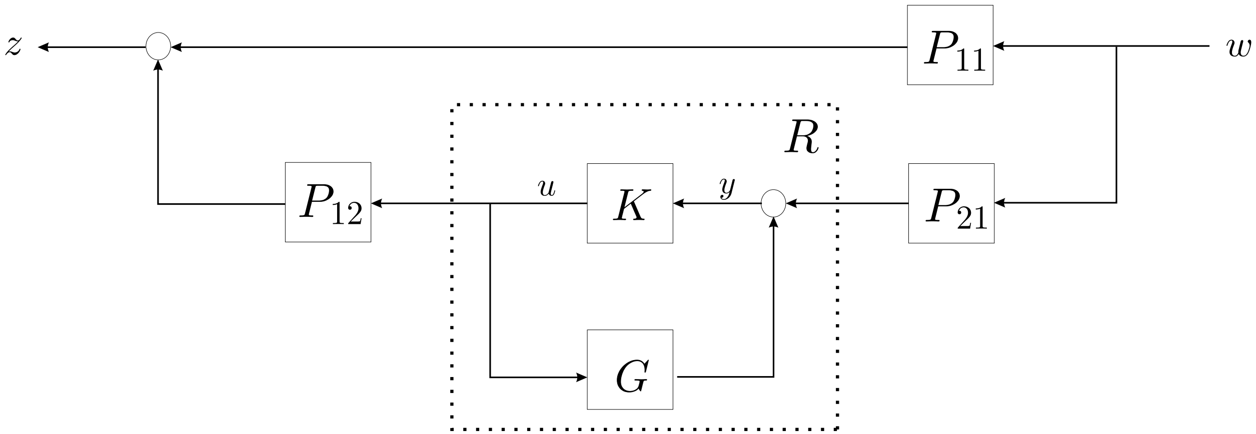

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Quadratic Invariance

The set S is called quadratically invariant with respect to G if

$$KGK \in S \qquad \text{for all } K \in S$$

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Main Result

S is quadratically invariant with respect to G if and only if

$$K \in S \qquad \Longleftrightarrow \qquad K(I - GK)^{-1} \in S$$

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Main Result

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Parameterization

$$\{K \mid K \text{ stabilizes } P,\ K \in S\} = \left\{R(I + GR)^{-1} \mid R \text{ stable},\ R \in S\right\}$$

Optimal Stabilizing Controller

Suppose $G \in \mathcal{R}_{sp}$ and $S \subseteq \mathcal{R}_p$.

We would like to solve

$$\begin{aligned} \text{minimize} \quad & \|P_{11} + P_{12}K(I - GK)^{-1}P_{21}\| \\ \text{subject to} \quad & K \text{ stabilizes } P \\ & K \in S \end{aligned}$$

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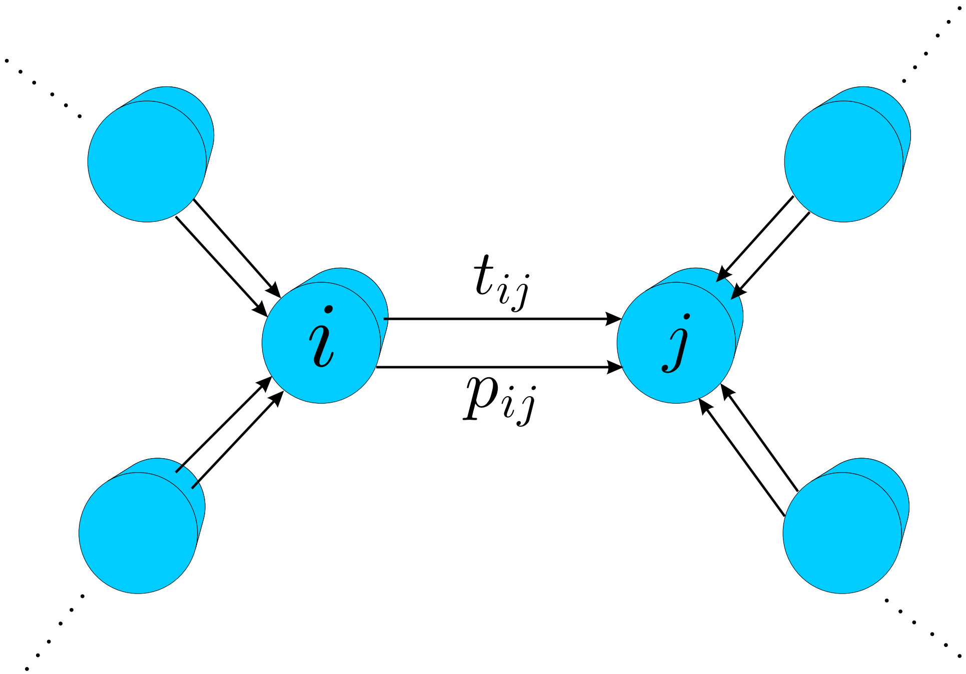
If S is quadratically invariant with respect to G, we may solve

$$\begin{aligned} \text{minimize} \quad & \|P_{11} + P_{12}RP_{21}\| \\ \text{subject to} \quad & R \text{ stable} \\ & R \in S \end{aligned}$$

which is convex.

Arbitrary Networks

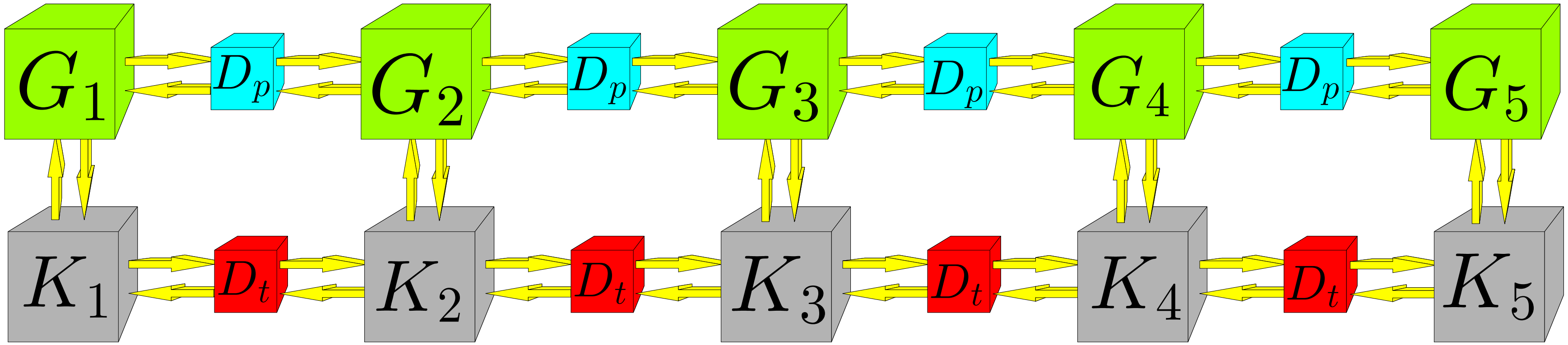


Assuming transmission delays satisfy the triangle inequality (i.e. transmissions take the quickest path)

S is quadratically invariant with respect to G if

$$t_{ij} \leq p_{ij} \qquad \text{for all } i, j$$

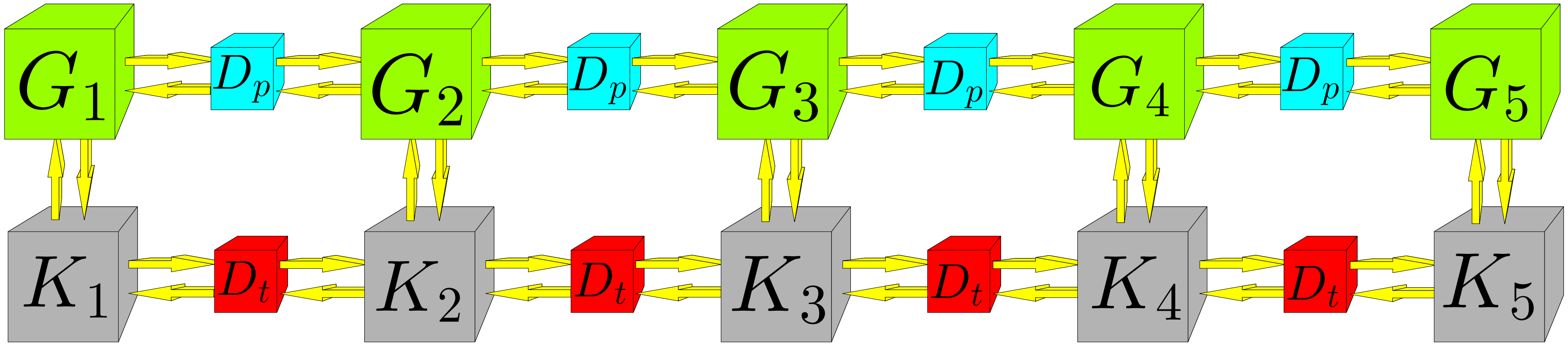
Distributed Control with Delays



- K_3 sees information from G_3 immediately
  G_2, G_4 after a delay of t
  G_1, G_5 after a delay of $2t$

- G_3 is affected by inputs from K_3 immediately
  K_2, K_4 after a delay of p
  K_1, K_5 after a delay of $2p$

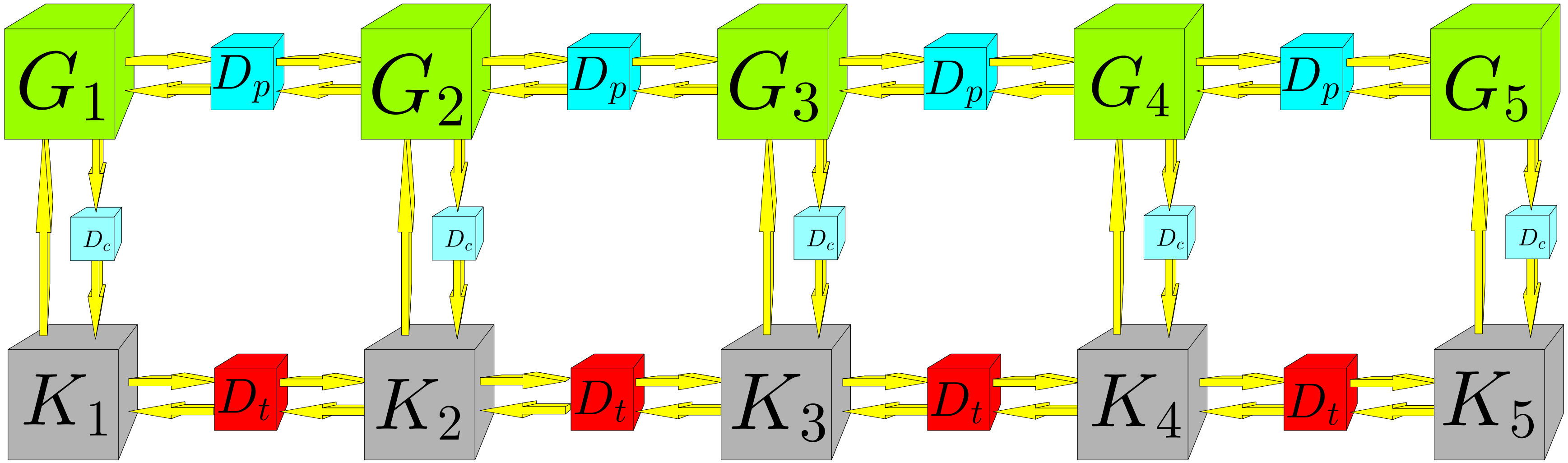
Distributed Control with Delays



S is quadratically invariant with respect to G if

$$t \leq p$$

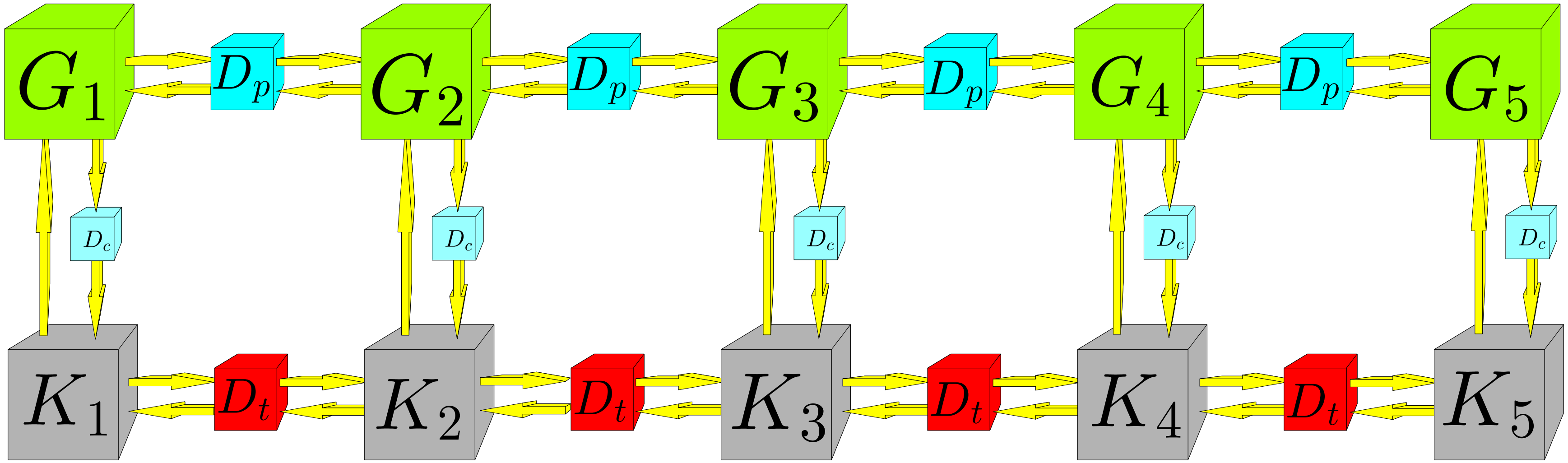
Distributed Control with Delays



- K_3 sees information from G_3 after a delay of c
  G_2, G_4 after a delay of $c + t$
  G_1, G_5 after a delay of $c + 2t$

- G_3 is affected by inputs from K_3 immediately
  K_2, K_4 after a delay of p
  K_1, K_5 after a delay of $2p$

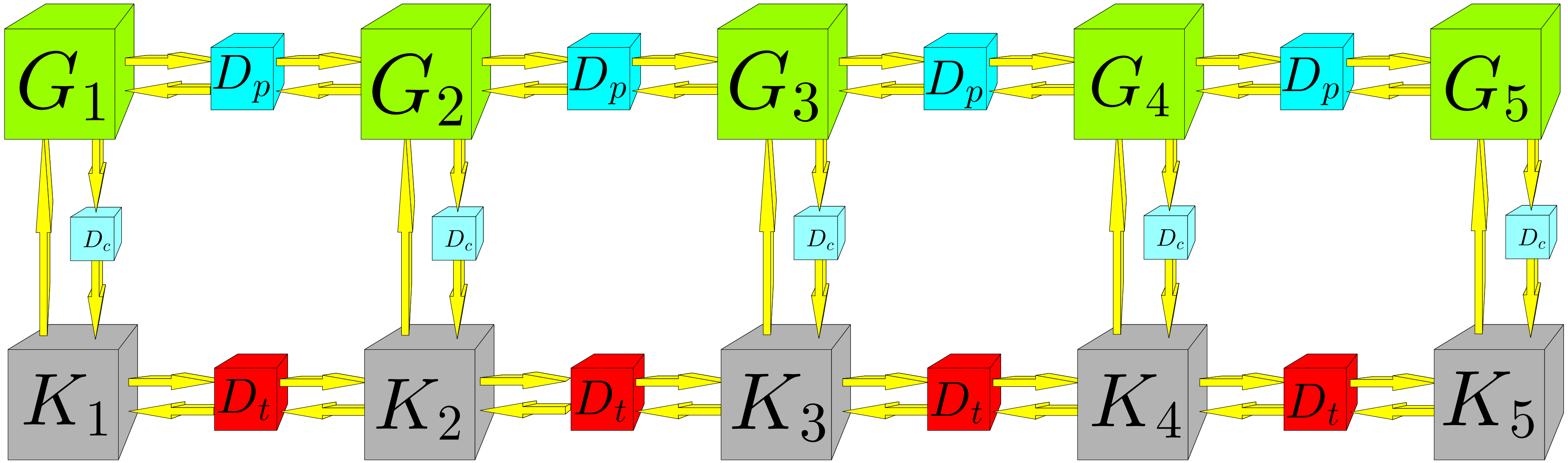
Distributed Control with Delays



Without computational delay,
S is quadratically invariant with respect to G if

$$t \leq p$$

Distributed Control with Delays



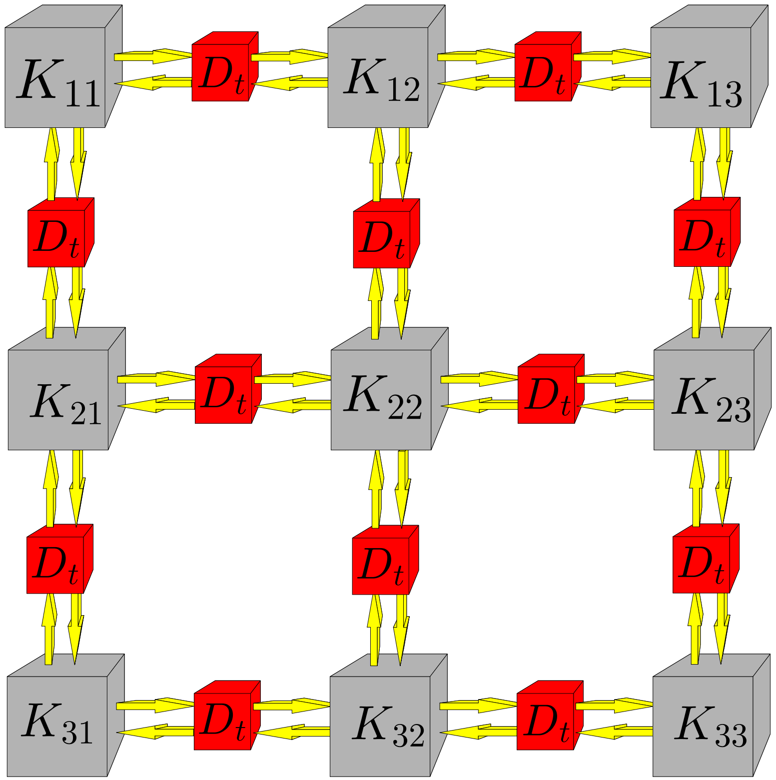
Without computational delay,
S is quadratically invariant with respect to G if

$$t \leq p$$

If computational delay is also present, then
S is quadratically invariant with respect to G if

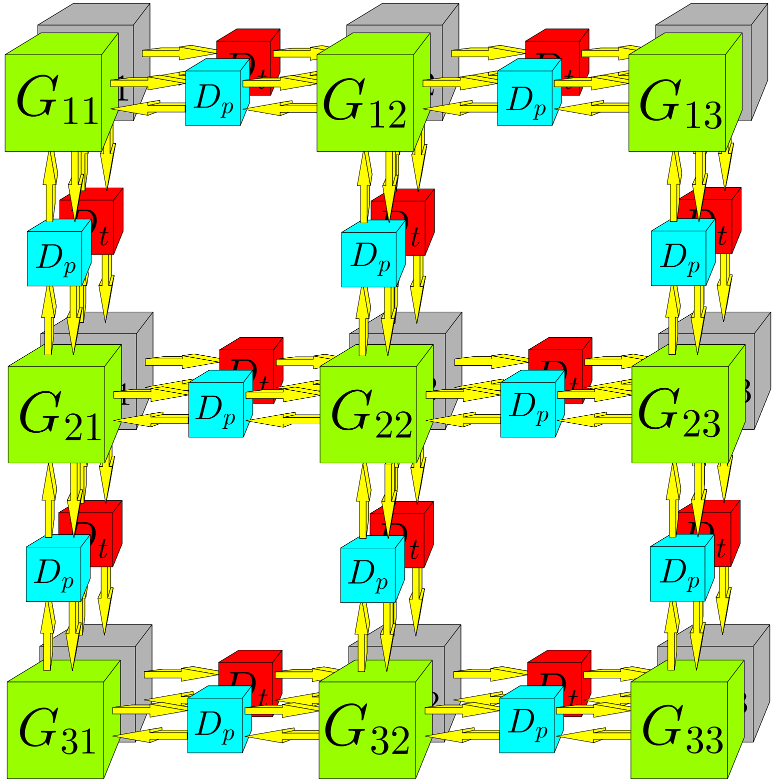
$$t \leq p + \frac{c}{n-1}$$

Two-Dimensional Lattice



Assuming controllers communicate along edges

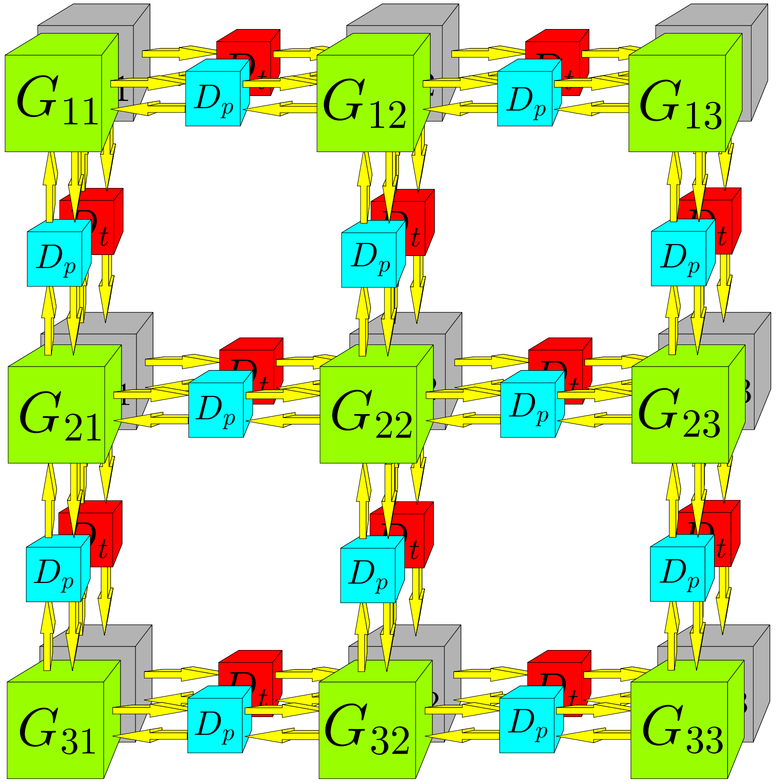
Two-Dimensional Lattice



Assuming controllers communicate along edges

Assuming dynamics propagate along edges

Two-Dimensional Lattice



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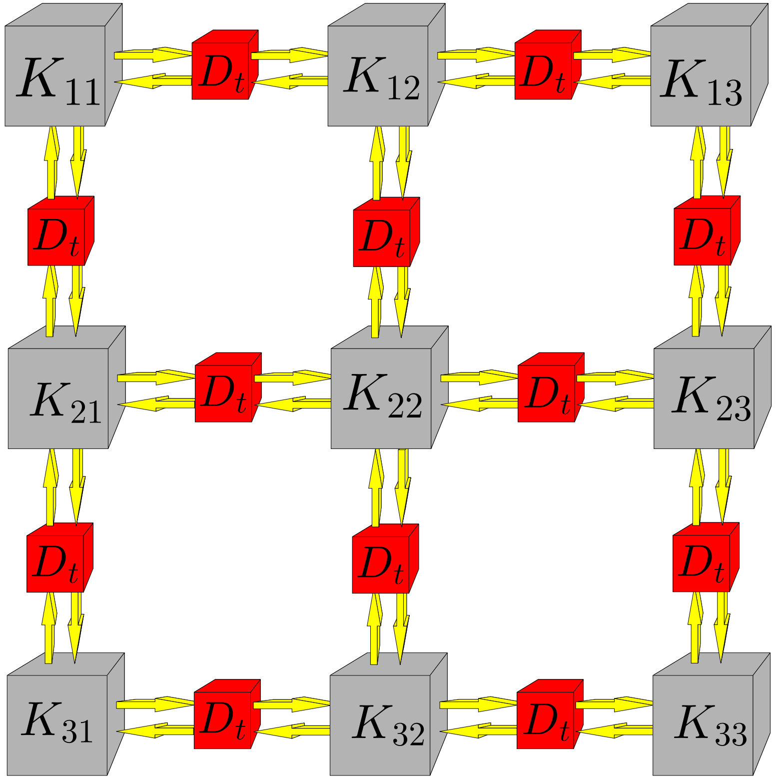
$$t \leq p$$

Two-Dimensional Lattice



Assuming controllers communicate along edges

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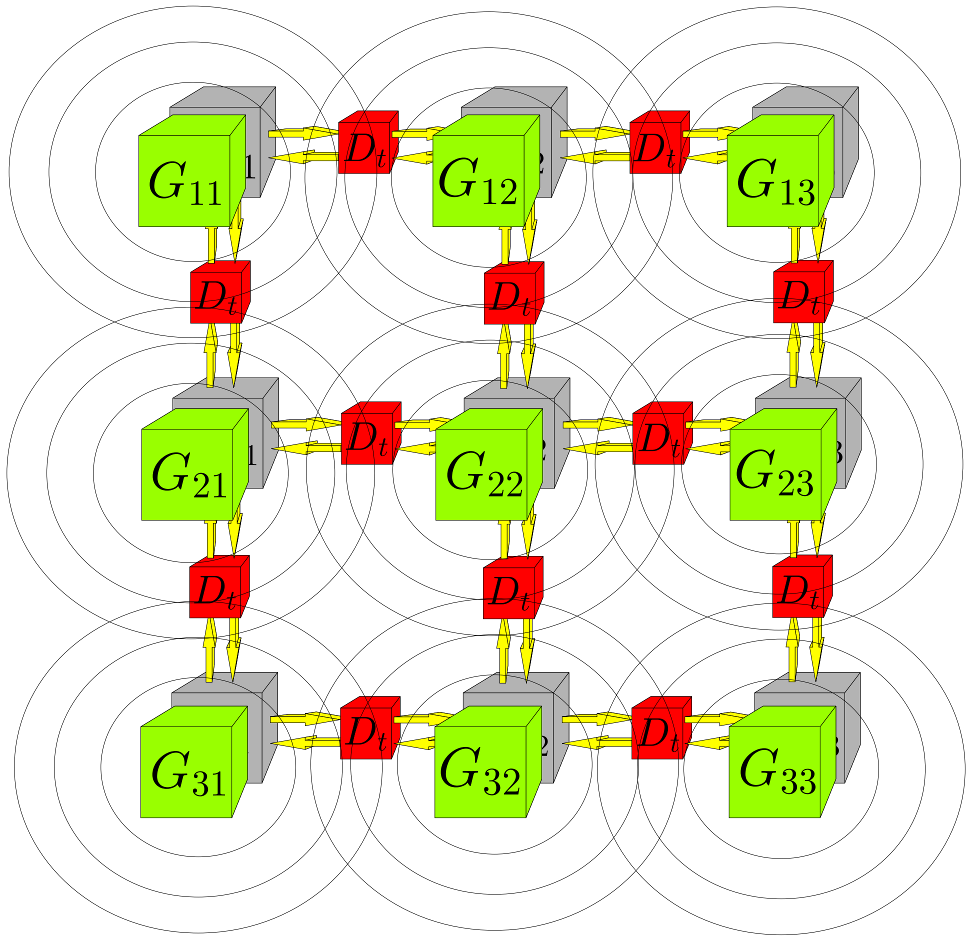
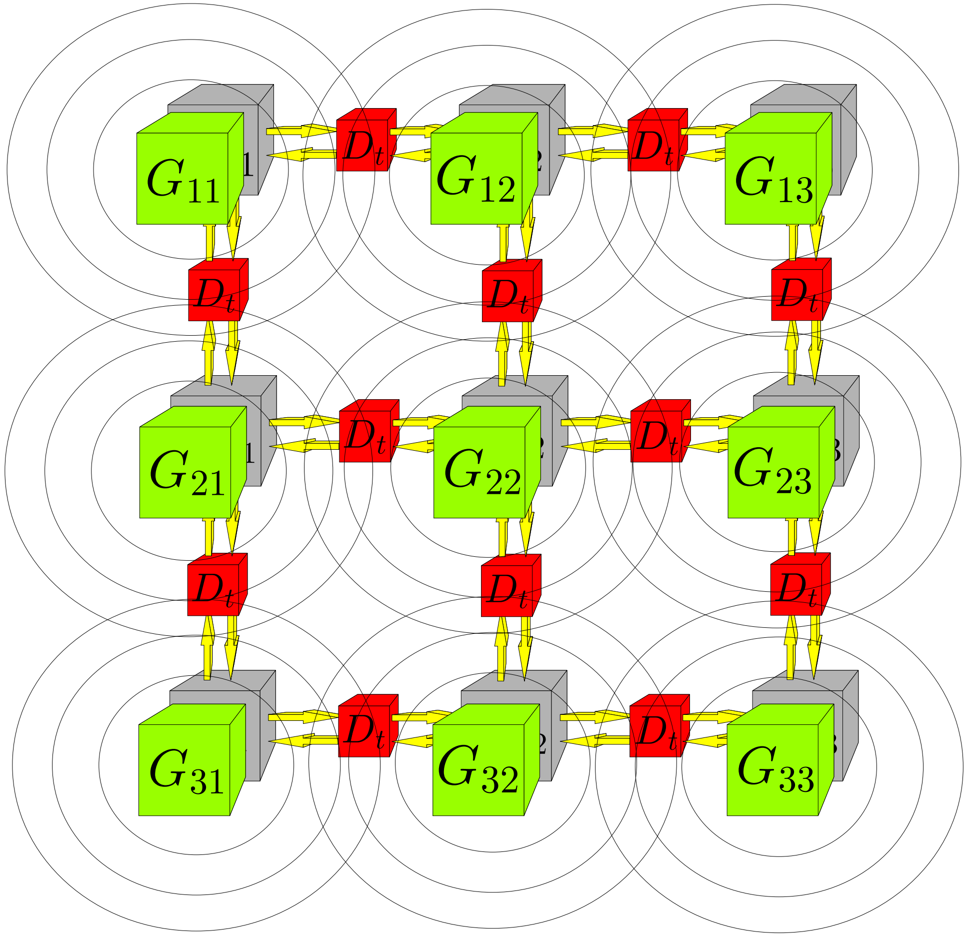


Assuming controllers communicate along edges

Assuming dynamics propagate outward
so that delay is proportional to geometric distance

Two-Dimensional Lattice



S is quadratically invariant with respect to G if

$$t \leq \frac{p}{\sqrt{2}}$$

Sparsity Example

Suppose

$$G \sim \begin{bmatrix} \bullet & \circ & \circ & \circ & \circ \\ \bullet & \bullet & \circ & \circ & \circ \\ \bullet & \bullet & \bullet & \circ & \circ \\ \bullet & \bullet & \bullet & \bullet & \circ \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{bmatrix} \qquad S = \left\{ K \;\middle|\; K \sim \begin{bmatrix} \circ & \circ & \circ & \circ & \circ \\ \circ & \bullet & \circ & \circ & \circ \\ \circ & \bullet & \circ & \circ & \circ \\ \bullet & \bullet & \bullet & \circ & \circ \\ \bullet & \bullet & \bullet & \circ & \bullet \end{bmatrix} \right\}$$

For arbitrary $K \in S$

$$KGK \sim \begin{bmatrix} \circ & \circ & \circ & \circ & \circ \\ \circ & \bullet & \circ & \circ & \circ \\ \circ & \bullet & \circ & \circ & \circ \\ \bullet & \bullet & \bullet & \circ & \circ \\ \bullet & \bullet & \bullet & \circ & \bullet \end{bmatrix}$$

Hence S is quadratically invariant with respect to G.

Same Structure Synthesis

Suppose

$$G \sim \begin{bmatrix} \circ & \circ & \circ \\ \bullet & \circ & \circ \\ \circ & \bullet & \bullet \end{bmatrix} \qquad S = \left\{ K \;\middle|\; K \sim \begin{bmatrix} \circ & \circ & \circ \\ \bullet & \circ & \circ \\ \circ & \bullet & \bullet \end{bmatrix} \right\}$$

Then

$$KGK \sim \begin{bmatrix} \circ & \circ & \circ \\ \circ & \circ & \circ \\ \bullet & \bullet & \bullet \end{bmatrix}$$

so S is *not* quadratically invariant with respect to G.

In fact,

$$K(I - GK)^{-1} \sim \begin{bmatrix} \circ & \circ & \circ \\ \bullet & \circ & \circ \\ \bullet & \bullet & \bullet \end{bmatrix}$$

Small Gain

If $|a| < 1$ then

$$(1-a)^{-1} \;=\; 1 + a + a^2 + a^3 + \dots$$

for example

$$\left(1-\frac{1}{2}\right)^{-1} \;=\; 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$$

more generally, if $\|A\| < 1$ then

$$(1-A)^{-1} \;=\; I + A + A^2 + A^3 + \dots$$

Not So Small Gain

Let

$$Q = I + A + A^2 + A^3 + \ldots$$

then

$$AQ = \quad A + A^2 + A^3 + \ldots$$

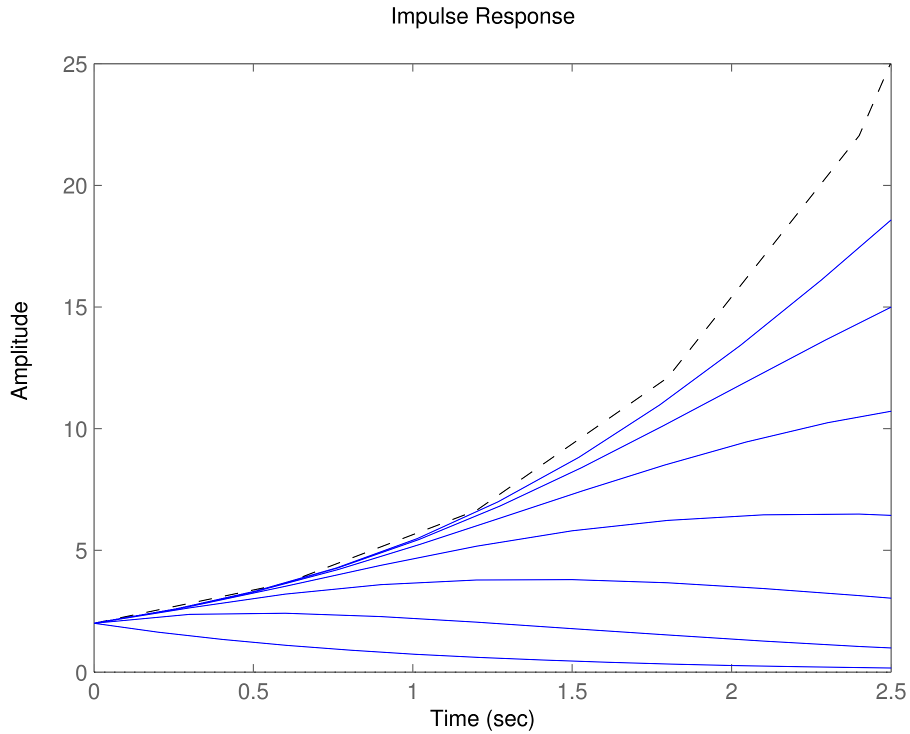
subtracting we get

$$(I - A)Q = I$$

and so

$$Q = (I - A)^{-1}$$

Convergence to Unstable Operator

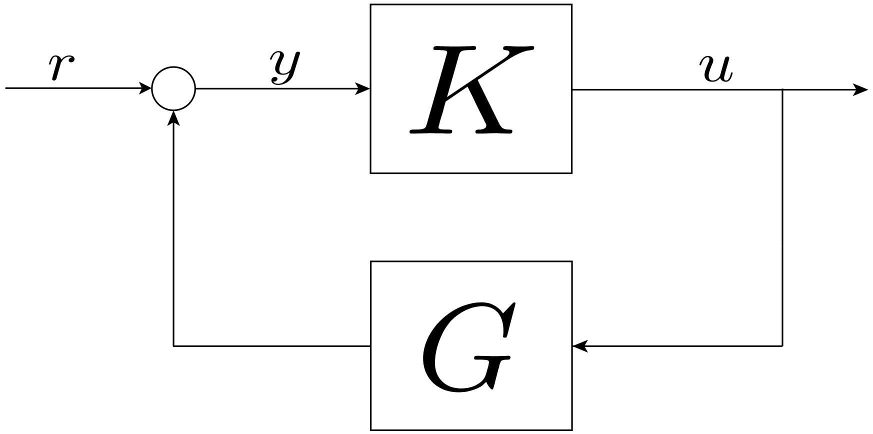


- Let $W(s) = \frac{2}{s+1}$
- Plot shows impulse response of $\sum_{k=0}^{N} W^k$ for $N = 1, \ldots, 7$
- Converges to that of $\frac{I}{I-W} = \frac{s+1}{s-1}$
- In what topology do the associated operators converge?

Conditions for Convergence: Inert

- Very broad, reasonable class of plants and controllers
- Basically, impulse response must be finite at any time T
- Arbitrarily large
- Includes the case $G \in \mathcal{R}_{sp},\ K \in \mathcal{R}_p$

Proof Sketch



$$KGK \in S \qquad \text{for all } K \in S$$

Suppose $K \in S$. Then

$$K(I - GK)^{-1} \; = \; K + KGK + K(GK)^2 + \ldots \;\; \in \;\; S$$

Consider Nonlinear

Let

$$Q = I + A + A^2 + A^3 + \ldots$$

then

$$AQ = \qquad A + A^2 + A^3 + \ldots$$

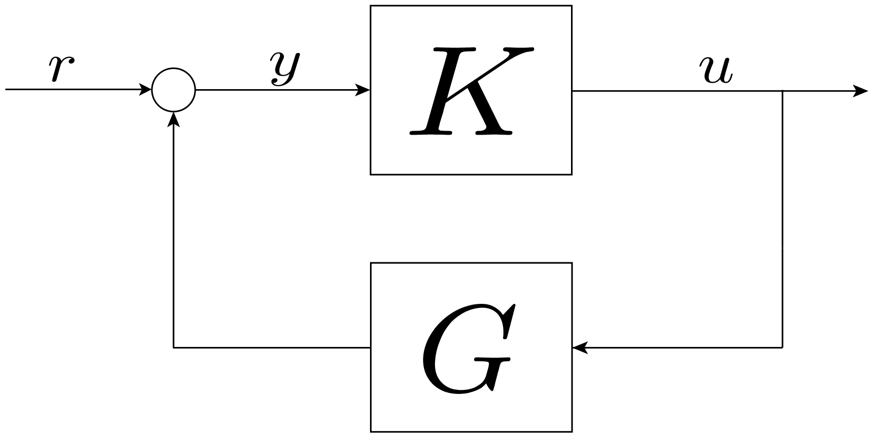
subtracting we get

$$(I - A)Q = I$$

and so

$$Q = (I - A)^{-1}$$

Block Diagram Algebra



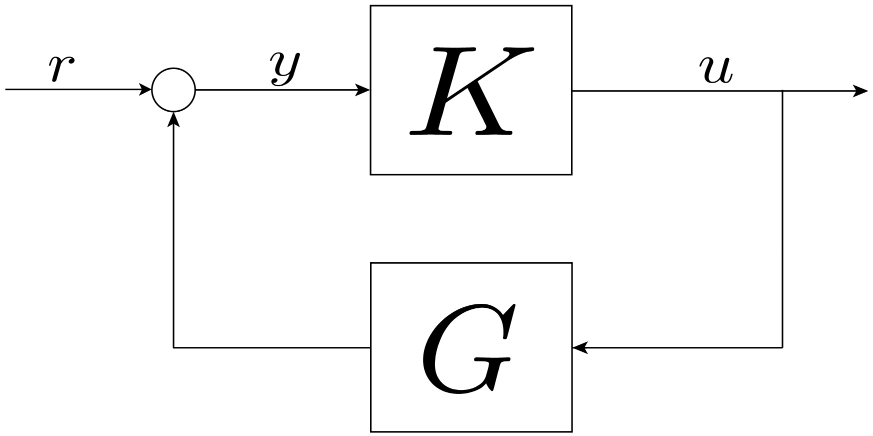
For a given r, we seek y, u such that

$$\begin{aligned} y &= r + Gu \\ u &= Ky \end{aligned}$$

and then define Y, R such that

$$\begin{aligned} y &= Yr = (I - GK)^{-1}r \\ u &= Rr = K(I - GK)^{-1}r \end{aligned}$$

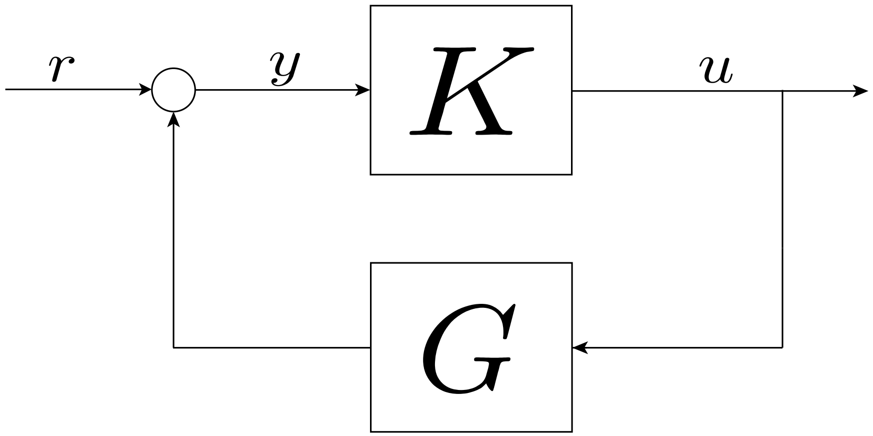
Iteration of Signals



We can define the following iteration, commensurate with the diagram

$$\begin{aligned} y^{(0)} &= r \\ u^{(n)} &= Ky^{(n)} \\ y^{(n+1)} &= r + Gu^{(n)} \end{aligned}$$

Iteration of Signals and Operators



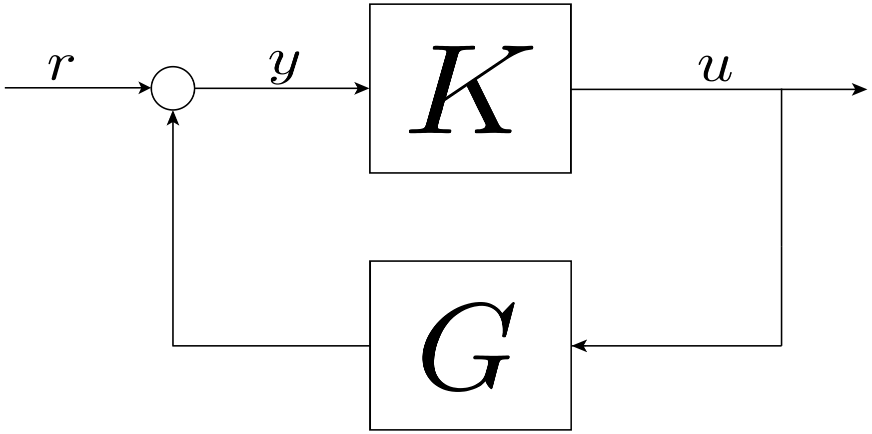
We can define the following iterations, commensurate with the diagram

$$y^{(0)} = r \qquad Y^{(0)} = I$$

$$u^{(n)} = Ky^{(n)} \qquad R^{(n)} = KY^{(n)}$$

$$y^{(n+1)} = r + Gu^{(n)} \qquad Y^{(n+1)} = I + GR^{(n)}$$

Iteration of Signals

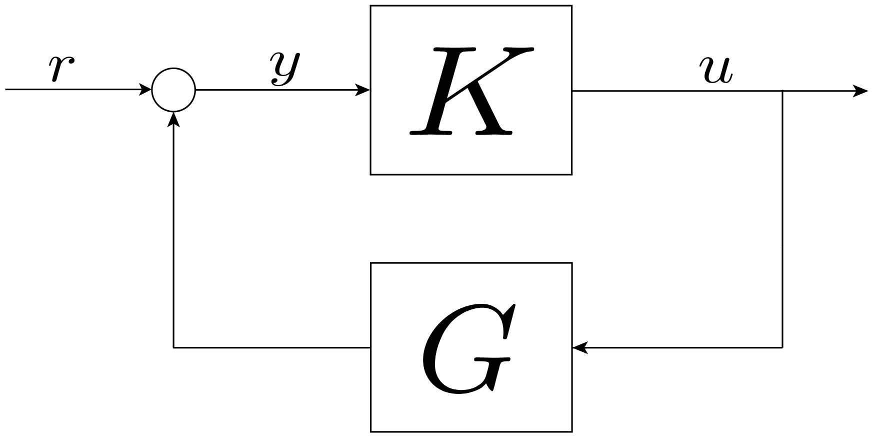


We then get the following recursions

$$y^{(0)} = r \qquad\qquad u^{(0)} = Kr$$

$$y^{(n+1)} = r + GKy^{(n)} \qquad\qquad u^{(n+1)} = K(r + Gu^{(n)})$$

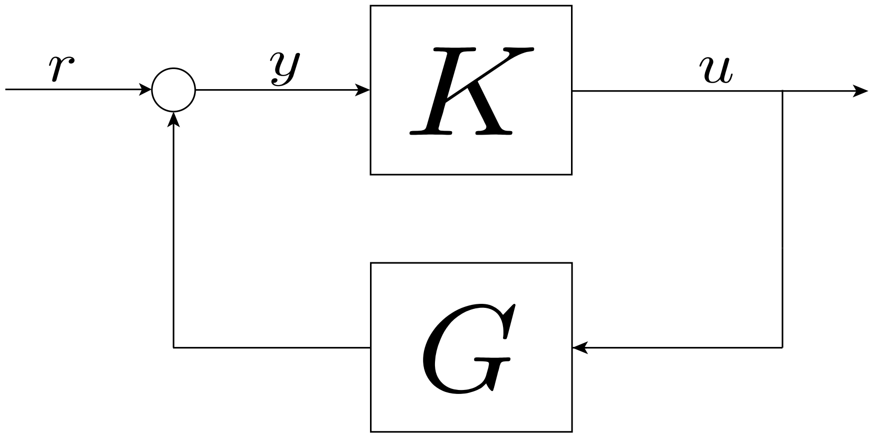
Iteration of Signals



We then get the following recursions

$$\begin{aligned} y^{(0)} &= r \\ y^{(n+1)} &= r + GKy^{(n)} \\ &\vdots \\ y &= \lim_{n\to\infty} y^{(n)} \end{aligned} \qquad \begin{aligned} u^{(0)} &= Kr \\ u^{(n+1)} &= K(r + Gu^{(n)}) \\ &\vdots \\ u &= \lim_{n\to\infty} u^{(n)} \end{aligned}$$

Iteration of Operators



We then get the following recursions

$$\begin{aligned} Y^{(0)} &= I \\ Y^{(n+1)} &= I + GKY^{(n)} \\ &\vdots \\ Y &= \lim_{n\to\infty} Y^{(n)} \end{aligned} \qquad \begin{aligned} R^{(0)} &= K \\ R^{(n+1)} &= K(I + GR^{(n)}) \\ &\vdots \\ R &= \lim_{n\to\infty} R^{(n)} \end{aligned}$$

Conditions for Convergence

- Very broad, reasonable class of plants and controllers
- Arbitrarily large
- Includes the case G strictly causal, K causal
- Includes the case $G \in \mathcal{R}_{sp},\ K \in \mathcal{R}_p$

Parameterization - Nonlinear

$$\{K \mid K \text{ stabilizes } G\} = \left\{R(I + GR)^{-1} \mid R \text{ stable}\right\}$$

- C.A. Desoer and R.W. Liu (1982)
- V. Anantharam and C.A. Desoer (1984)

New Invariance Condition

$$K_1(I \pm GK_2) \in S \qquad \text{for all } K_1, K_2 \in S$$

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Invariance under Feedback

If this condition is satisfied, then

$$K \in S \qquad \Longleftrightarrow \qquad K(I - GK)^{-1} \in S$$

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Invariance under Feedback

If this condition is satisfied, then

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Parameterization

$$\{K \mid K \text{ stabilizes } G,\ K \in S\} = \left\{R(I + GR)^{-1} \mid R \text{ stable},\ R \in S\right\}$$

Proof Sketch

$$K_1(I \pm GK_2) \in S \qquad \forall\, K_1, K_2 \in S$$

Suppose that $K \in S$.

Then $R^{(0)} = K \in S$.

If we assume that $R^{(n)} \in S$, then

$$R^{(n+1)} \;=\; K(I + GR^{(n)}) \;\in\; S$$

Thus $R^{(n)} \in S$ for all $n \in \mathbb{Z}_+$.

$$R \;=\; \lim_{n \to \infty} R^{(n)} \in S$$

Conclusions

- Quadratic invariance allows parameterization of all (LTI) stablizing decentralized controllers.
- Similar condition allows parameterization of all (NLTV) stabilizing decentralized controllers
- These condition are satisfied when communications are faster than the propagation of dynamics.

Quadratic Invariance

The set S is called quadratically invariant with respect to G if

$$KGK \in S \qquad \text{for all } K \in S$$

?????????

$$K_1(I \pm GK_2) \in S \qquad \text{for all } K_1, K_2 \in S$$

Open Questions / Future Work

- Unstable plant
- Is there a weaker condition which achieves the same results? Perhaps

$$K(I \pm GK) \in S \qquad \forall\, K \in S\ ?$$

- When is the optimal (possibly nonlinear) controller linear?
- When (else) does it hold?
- What should we call it?