On the optimality of localized distributed controllers

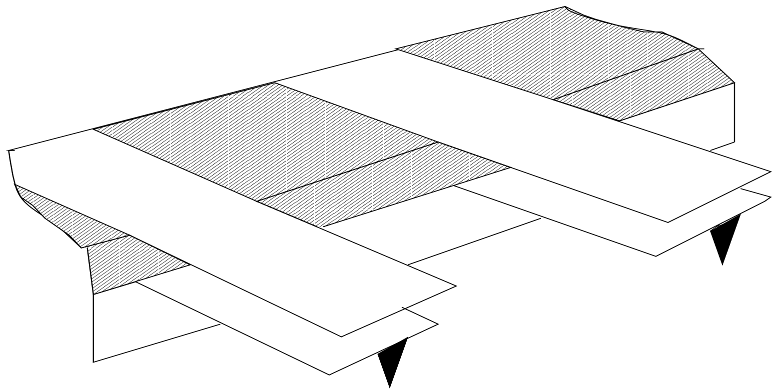
Mihailo Jovanović
www.umn.edu/∼mihailo

ELECTRICAL AND COMPUTER ENGINEERING
UNIVERSITY OF MINNESOTA

CDC-ECC'05 Workshop – December 11, 2005

Motivating example

ELECTROSTATICALLY ACTUATED MICRO-CANTILEVERS (NAPOLI ET. AL.):



POTENTIAL APPLICATION: MASSIVELY PARALLEL DATA STORAGE & RETRIEVAL

problem: slow scans $\equiv$ low throughput

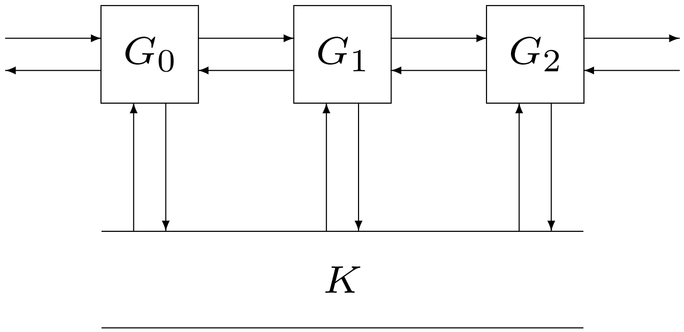
solution: go massively parallel

ISSUES:

tightly coupled dynamics	$\Rightarrow$	spatio-temporal instabilities
large arrays $\approx$ 10000 devices	$\Rightarrow$	localized control imperative

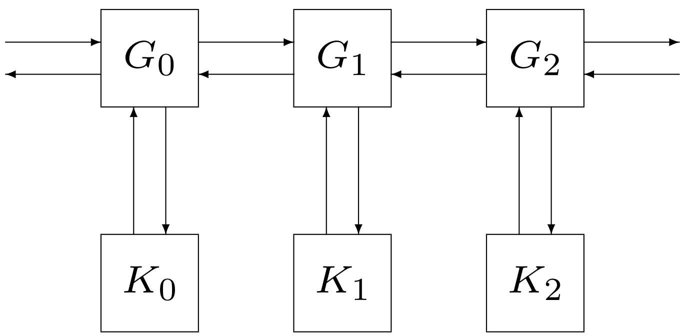
Controller architectures

CENTRALIZED:



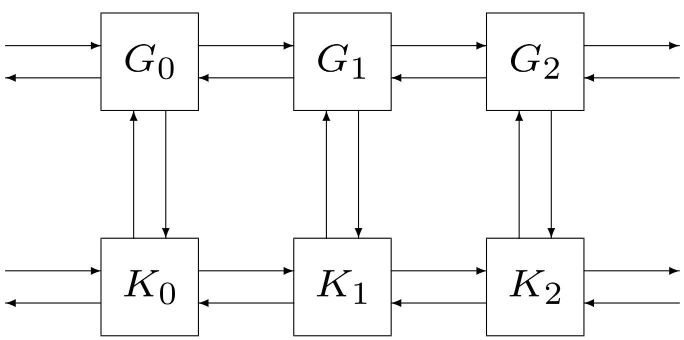
BEST PERFORMANCE
EXCESSIVE COMMUNICATION

FULLY DECENTRALIZED:



WORST PERFORMANCE
NO COMMUNICATION

LOCALIZED:



MANY POSSIBLE ARCHITECTURES

Motivation

Optimal design with *a priori* assigned localization constraints: difficult problem
(Voulgaris & Bamieh, Langbort & D'Andrea, Rotkowitz & Lall, Castro & Paganini)

Alternative problem:
given a stabilizing localized controller, is it optimal (in the LQR sense)?

The inverse problem of optimal control (Kalman '64)

Traditional motivation: optimality $\Rightarrow$ robustness

Additional motivation: insight about spatial extent of the LQR weights

Outline

❶ CLASS OF SYSTEMS

⋆ spatially invariant systems

❷ EXAMPLES OF OPTIMAL DISTRIBUTED DESIGN

⋆ fully decentralized performance indices $\Rightarrow$ centralized controllers

❸ THE INVERSE PROBLEM OF OPTIMAL DISTRIBUTED CONTROL

⋆ return difference inequality

❹ EXAMPLES OF INVERSELY OPTIMAL DISTRIBUTED DESIGN

⋆ departure from fully decentralized performance indices

❺ CONCLUDING REMARKS

Spatially invariant systems

Bamieh, Paganini, Dahleh '02

$$\partial_t \psi(t,\xi) \;=\; \mathcal{A}\psi(t,\xi) \;+\; \mathcal{B}u(t,\xi)$$

spatial coordinate: $\xi \;\in\; \mathbb{G}$

spatial invariant operators: $\mathcal{A}$, $\mathcal{B}$

Spatial Fourier transform

$$\dot{\hat{\psi}}_\kappa(t) \;=\; \hat{\mathcal{A}}_\kappa \hat{\psi}_\kappa(t) \;+\; \hat{\mathcal{B}}_\kappa \hat{u}_\kappa(t)$$

spatial frequency: $\kappa \;\in\; \hat{\mathbb{G}}$

multiplication operators: $\hat{\mathcal{A}}_\kappa,\; \hat{\mathcal{B}}_\kappa$

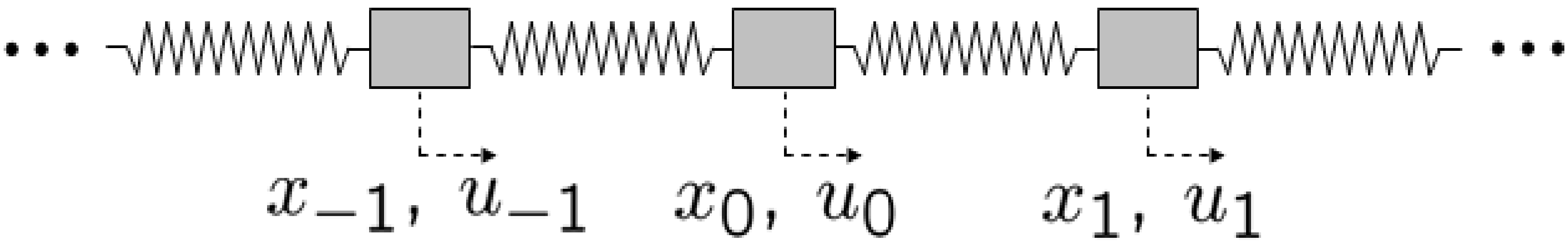
HEAT EQUATION OVER $\mathbb{G} := \mathbb{R}$

$$\psi_t(t,\xi) \;=\; \psi_{\xi\xi}(t,\xi) \;+\; c\psi(t,\xi) \;+\; u(t,\xi), \quad \xi \in \mathbb{R}$$

$$\Updownarrow$$

$$\dot{\hat{\psi}}_\kappa(t) \;=\; (c \,-\, \kappa^2)\hat{\psi}_\kappa(t) \;+\; \hat{u}_\kappa(t), \quad \kappa \in \mathbb{R}$$

MASS-SPRING SYSTEM OVER $\mathbb{G} := \mathbb{Z}$



$$\dot{\psi}(t,\xi) \;=\; \begin{bmatrix} 0 & 1 \\ T_{-1} \,-\, 2 \,+\, T_1 & 0 \end{bmatrix} \psi(t,\xi) \;+\; \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t,\xi), \quad \xi \in \mathbb{Z}$$

$$\Updownarrow$$

$$\dot{\hat{\psi}}_\kappa(t) \;=\; \begin{bmatrix} 0 & 1 \\ 2(\cos\,\kappa \,-\, 1) & 0 \end{bmatrix} \hat{\psi}_\kappa(t) \;+\; \begin{bmatrix} 0 \\ 1 \end{bmatrix} \hat{u}_\kappa(t), \quad \kappa \in [0,\, 2\pi)$$

Distributed LQR design

$$\begin{aligned} \partial_t \psi(t,\xi) &= \mathcal{A}\psi(t,\xi) + \mathcal{B}u(t,\xi) \\ J &= \frac{1}{2}\int_0^\infty \left(\langle \psi, \mathcal{Q}\psi\rangle + \langle u, \mathcal{R}u\rangle\right) \mathrm{d}t \end{aligned}$$

$$\Updownarrow$$

$$\begin{aligned} \dot{\hat{\psi}}_\kappa(t) &= \hat{\mathcal{A}}_\kappa \hat{\psi}_\kappa(t) + \hat{\mathcal{B}}_\kappa \hat{u}_\kappa(t) \\ J &= \frac{1}{2}\int_0^\infty \int_{\hat{\mathbb{G}}} \left(\hat{\psi}_\kappa^*(t)\hat{\mathcal{Q}}_\kappa\hat{\psi}_\kappa(t) + \hat{u}_\kappa^*(t)\hat{\mathcal{R}}_\kappa\hat{u}_\kappa(t)\right) \mathrm{d}\kappa\, \mathrm{d}t \\ \hat{u}_\kappa(t) &:= \hat{\mathcal{K}}_\kappa\hat{\psi}_\kappa(t) = -\hat{\mathcal{R}}_\kappa^{-1}\hat{\mathcal{B}}_\kappa^*\hat{\mathcal{P}}_\kappa\hat{\psi}_\kappa(t), \quad \kappa \in \hat{\mathbb{G}} \end{aligned}$$

HEAT EQUATION

$$\mathcal{Q} := qI, \;\; \mathcal{R} := rI \quad \Rightarrow \quad \hat{\mathcal{K}}_\kappa \; = \; -\,(c - \kappa^2) \, - \, \sqrt{(c - \kappa^2)^2 + q/r}$$

$\hat{\mathcal{K}}_\kappa$ - irrational function of κ $\Rightarrow$ can't be implemented by a localized controller

$$u(t,\xi) \; = \; \int_{\mathbb{R}} \mathcal{K}(\xi - \zeta)\psi(t,\zeta)\,\mathrm{d}\zeta, \quad \xi \in \mathbb{R}$$

MASS-SPRING SYSTEM

$$\begin{array}{rcl} \mathcal{Q} & := & \begin{bmatrix} q_1 I & 0 \\ 0 & q_2 I \end{bmatrix} \\ \mathcal{R} & := & rI \end{array} \quad \Rightarrow \quad \begin{array}{rcl} \hat{\mathcal{K}}_\kappa & := & \begin{bmatrix} \hat{\mathcal{K}}_{1\kappa} & \hat{\mathcal{K}}_{2\kappa} \end{bmatrix} \\ \hat{\mathcal{K}}_{1\kappa} & = & 2(1 - \cos\,\kappa) - \sqrt{4(\cos\,\kappa - 1)^2 + q_1/r} \\ \hat{\mathcal{K}}_{2\kappa} & = & -\sqrt{-2\hat{\mathcal{K}}_{1\kappa} + q_2/r} \end{array}$$

$\hat{\mathcal{K}}_\kappa$ - irrational function of κ $\Rightarrow$ can't be implemented by a localized controller

$$u(t,\xi) \; = \; \sum_{\zeta \in \mathbb{Z}} \mathcal{K}(\xi - \zeta)\psi(t,\zeta), \quad \xi \in \mathbb{Z}$$

Inverse problem of optimal distributed control

Given stabilizing $\mathcal{K}$, is it optimal wrt some $(\mathcal{Q}, \mathcal{R})$?

$\mathcal{K}$ - localized distributed controller

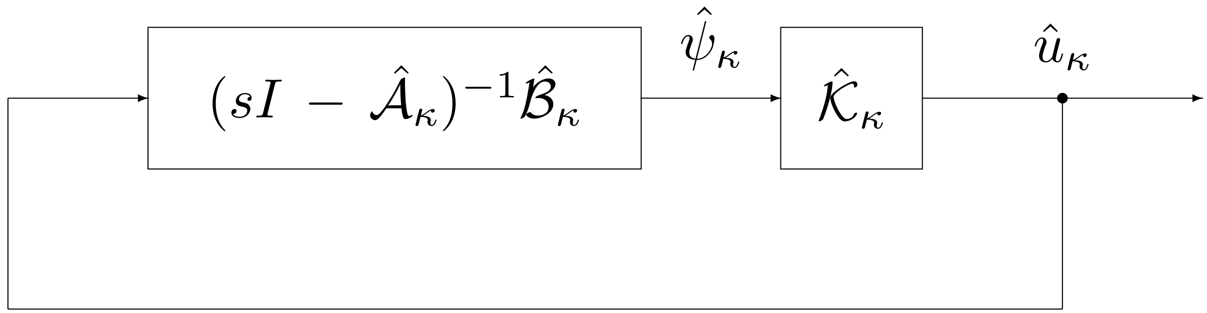
HEAT EQUATION

$$\begin{aligned} \psi_t(t,\xi) &= \psi_{\xi\xi}(t,\xi) + c\psi(t,\xi) + u(t,\xi) \\ u(t,\xi) &= -(c+\alpha)\psi(t,\xi) \\ \text{stability} &\Leftrightarrow \alpha > 0 \end{aligned}$$

MASS-SPRING SYSTEM

$$\begin{aligned} \dot{\psi}(t,\xi) &= \begin{bmatrix} 0 & 1 \\ T_{-1} - 2 + T_1 & 0 \end{bmatrix} \psi(t,\xi) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t,\xi) \\ u(t,\xi) &= -\begin{bmatrix} \alpha & \gamma \end{bmatrix} \psi(t,\xi) \\ \text{stability} &\Leftrightarrow \alpha, \gamma > 0 \end{aligned}$$

Conditions for inverse optimality



Sensitivity function

$$\hat{\mathcal{S}}_\kappa(s) \;:=\; \left(I \;-\; \hat{\mathcal{K}}_\kappa(sI \;-\; \hat{\mathcal{A}}_\kappa)^{-1}\hat{\mathcal{B}}_\kappa\right)^{-1}$$

Single input systems ($\hat{u}_\kappa \in \mathbb{C}$)

optimality

$$\Updownarrow$$

$$|\hat{\mathcal{S}}_\kappa(j\omega)| \;\leq\; 1, \;\; \forall\, \omega \in \mathbb{R}, \;\; \forall\, \kappa \in \hat{\mathbb{G}}$$

If satisfied, determine $\hat{\mathcal{Q}}_\kappa$ and $\hat{\mathcal{R}}_\kappa$ from

$$\hat{\mathcal{B}}_\kappa^*(j\omega I \,+\, \hat{\mathcal{A}}_\kappa^*)^{-1}\hat{\mathcal{Q}}_\kappa(j\omega I \,-\, \hat{\mathcal{A}}_\kappa)^{-1}\hat{\mathcal{B}}_\kappa \;=\; \hat{\mathcal{R}}_\kappa\left(1 \;-\; 1/|\hat{\mathcal{S}}_\kappa(j\omega)|^2\right)$$

Examples of inversely optimal design

HEAT EQUATION WITH FULLY DECENTRALIZED CONTROLLER

$$\begin{array}{rcl} \psi_t(t,\xi) & = & \psi_{\xi\xi}(t,\xi) \;+\; c\psi(t,\xi) \;+\; u(t,\xi) \\ u(t,\xi) & = & -\,(c\,+\,\alpha)\psi(t,\xi) \\ \text{stability} & \Leftrightarrow & \alpha \,>\, 0 \\ \text{optimality} & \Leftrightarrow & \alpha \,\geq\, c \end{array}$$

$$\mathcal{R} \;:=\; I \quad \Rightarrow \quad \mathcal{Q} \;=\; (\alpha^2 \,-\, c^2)I \;-\; 2(c\,+\,\alpha)\partial_{\xi\xi}$$

$$\Updownarrow$$

$$J \;=\; \frac{1}{2}\int_0^\infty \big((\alpha^2 \,-\, c^2)\,\langle\psi,\psi\rangle \;+\; 2(c\,+\,\alpha)\,\langle\psi_\xi,\psi_\xi\rangle \;+\; \langle u,u\rangle\,\big)\,\mathrm{d}t$$

MASS-SPRING SYSTEM WITH FULLY DECENTRALIZED CONTROLLER

$$\begin{aligned} \dot{\psi}(t,\xi) &= \begin{bmatrix} 0 & 1 \\ T_{-1} - 2 + T_1 & 0 \end{bmatrix} \psi(t,\xi) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t,\xi) \\ u(t,\xi) &= - \begin{bmatrix} \alpha & \gamma \end{bmatrix} \psi(t,\xi) \end{aligned}$$

$$\text{stability} \;\Leftrightarrow\; \alpha > 0, \;\; \gamma > 0$$

$$\text{optimality} \;\Leftrightarrow\; \alpha > 0, \;\; \gamma > \sqrt{2\alpha}$$

$$J = \frac{1}{2}\int_0^\infty \sum_{n\,\in\,\mathbb{Z}} \sum_{m\,\in\,\mathbb{Z}} \big(\psi^*(t,n) Q_{n-m} \psi(t,m) + u^*(t,n) R_{n-m} u(t,m)\big)\, \mathrm{d}t$$

$$R_0 = 1 \;\Rightarrow\; Q_0 = \begin{bmatrix} \alpha^2 + 4\alpha & 0 \\ 0 & \gamma^2 - 2\alpha \end{bmatrix}, \quad Q_{\pm 1} = \begin{bmatrix} -2\alpha & 0 \\ 0 & 0 \end{bmatrix}$$

Examples: summary

Heat equation

$$
\begin{aligned}
\psi_t(t,\xi) &= \psi_{\xi\xi}(t,\xi) + c\psi(t,\xi) + u(t,\xi) \\
J &= \frac{1}{2}\int_0^\infty \left(q_1 \langle \psi, \psi \rangle + q_2 \langle \psi_\xi, \psi_\xi \rangle + \langle u, u \rangle \right) \mathrm{d}t \\
q_2 &= 2\left(c + \sqrt{q_1 + c^2}\right) \;\Rightarrow\; \text{fully decentralized controller}
\end{aligned}
$$

Mass-spring system

$$
\begin{aligned}
\begin{bmatrix} \dot{\psi}_1(t,n) \\ \dot{\psi}_2(t,n) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ T_{-1} - 2 + T_1 & 0 \end{bmatrix} \begin{bmatrix} \psi_1(t,n) \\ \psi_2(t,n) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t,n) \\
J &= \frac{1}{2}\int_0^\infty \sum_{n\,\in\,\mathbb{Z}} \left(q_1\psi_1^2(t,n) + q_2\psi_2^2(t,n) + u^2(t,n) \right) \mathrm{d}t + \\
&\quad \frac{1}{2}\int_0^\infty \sum_{n\,\in\,\mathbb{Z}} q_3\, \psi_1(t,n) \left(\psi_1(t,n-1) + \psi_1(t,n+1) \right) \mathrm{d}t \\
q_3 &= 2\left(2 - \sqrt{4 + q_1}\right) \;\Rightarrow\; \text{fully decentralized controller}
\end{aligned}
$$

Optimal control of vehicular platoons

$$\begin{bmatrix} \dot{\psi}_1 \\ \dot{\psi}_2 \end{bmatrix} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} u, \quad u = -\left((aI + bL)\psi_1 + c\psi_2\right)$$

$$L := \begin{bmatrix} 1 & -1 & 0 & & 0 & 0 & 0 \\ -1 & 2 & -1 & & 0 & 0 & 0 \\ 0 & -1 & 2 & & 0 & 0 & 0 \\ & & & \ddots & & & \\ 0 & 0 & 0 & & 2 & -1 & 0 \\ 0 & 0 & 0 & & -1 & 2 & -1 \\ 0 & 0 & 0 & & 0 & -1 & 1 \end{bmatrix}$$

INVERSE OPTIMALITY WRT:

$$J = \frac{1}{2}\int_0^\infty (\psi_1^* Q_1 \psi_1 + \psi_2^* Q_2 \psi_2 + r u^* u)\, \mathrm{d}t, \quad r > 0$$

$$Q_1 = r\,(aI + bL)^2, \quad Q_2 = r\left((c^2 - 2a)I - 2bL\right)$$

$$a > 0, \quad b > 0, \quad c \geq \sqrt{2(a + b\lambda_1(L))}, \quad \lambda_1(L) = 2\left(1 - \cos\frac{(M-1)\pi}{M}\right)$$

Concluding remarks

- INVERSE OPTIMAL DISTRIBUTED DESIGN FOR SPATIALLY INVARIANT SYSTEMS

  Jovanović, ACC'05:

  ⋆ Frequency domain condition for optimality

  ⋆ Return difference inequality

  ⋆ Departure from fully decentralized performance indices

- ONGOING RESEARCH

  ⋆ Parametrization of all LQR weights

  ⋆ A design tool?

  ⋆ Optimal design for other classes of distributed control problems

Additional info

Mihailo Jovanović
www.umn.edu/∼mihailo