Due Friday 10/13/06 at the beginning of the class

Your name:

Your Student ID:

Important points:

- This is a take-home, open lecture notes exam.
- You are not allowed to talk to anybody about this exam until you hand it in.
- Work on the exam for no more than three hours.
- Think before you start solving problems!
- Explain your solutions clearly and provide all necessary details.
- Good luck!

Problem 1: Consider the system described by the following differential equation

$$\ddot{y} + (1 + y)\dot{y} - 2y + 0.5y^3 = u.$$

1. Rewrite the system in a state space form.
2. Determine the equilibrium points of the *unforced* system.
3. Use Lyapunov indirect method to study stability properties of each equilibrium point.
4. Determine the impulse response of the system linearized around the origin.

Problem 2: Consider the second-order nonlinear state-space system

$$\begin{aligned} \dot{x}_1 &= -x_1 - x_2 + x_1 x_2, \\ \dot{x}_2 &= x_1 - x_1^2. \end{aligned}$$

1. What are the equilibrium points of the system?
2. What is the linearized model around the equilibrium point at the origin, and what does the linearized model tell you about the stability of this equilibrium point?
3. By analyzing how the function $(x_1^2 + x_2^2)/2$ behaves along trajectories of the nonlinear system, deduce as much as you can about the stability properties of the equilibrium point at the origin.

Problem 3: Consider the dynamical system $\dot{x} = Ax$ with A a 2×2 matrix. Suppose that

$$x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad x(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad x(2) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

1. Find e^A.
2. Suppose a new initial condition

$$x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

   is applied to the system, find $x(1)$ and $x(2)$.
3. Can you determine matrix A from the above?

Problem 4: Let A be a *symmetric* $n \times n$ matrix with all its eigenvalues in the open left half of the complex plane, and Q be a positive definite matrix.

1. Suppose P satisfies the Lyapunov equation $A^T P + PA = -I$. What is a solution P as a function of A? Is it unique?

2. Suppose P satisfies the Lyapunov equation $A^T P + PA = -Q$. Show that $\text{trace}(P) = -\frac{1}{2}\,\text{trace}(A^{-1}Q)$.

Problem 5: Consider the dynamical system

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= x_1 - \operatorname{sat}(2x_1 + x_2),\end{aligned}$$

where sat$(\cdot)$ refers to the saturation function defined as

$$\operatorname{sat}(y) = \begin{cases} -1 & \text{for} \quad y < -1,\\ y & \text{for} \quad |y| \le 1,\\ 1 & \text{for} \quad y > 1.\end{cases}$$

1. Is the origin asymptotically stable?
2. Is the origin *globally* asymptotically stable?