Due Monday 12/04/06 at the beginning of the class

Your name:

Your Student ID:

Important points:

- This is a take-home, open lecture notes exam. You are not allowed to use any textbooks or look at your homework assignments.
- You are not allowed to talk to anybody about this exam until you hand it in.
- Work on the exam as much as you want to.
- Think before you start solving problems!
- You can use MATLAB for your calculations/computations/simulations.
- Explain your solutions clearly and provide all necessary details including the listings of your MATLAB scripts.
- Good luck!

Problem 1: A schematic of an electrohydraulic servo system is shown in Figure 1. A mathematical model of this system is derived using Newton's Second Law for the rotational motion of the motor shaft, the continuity equation for each chamber of the hydraulic motor, and by approximating the connection between the torque motor and the first stage of the electrohydraulic servovalve by a first order transfer function. This representation is given by

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2, \\ \dot{x}_2 &= -x_1 - x_2 + x_3\sqrt{10 - x_2}, \\ \dot{x}_3 &= -x_3 + u, \end{aligned} \tag{1}$$

where x_1 is motor angular velocity, x_2 is load pressure differential, x_3 is servo-valve displacement, and u is voltage control signal.

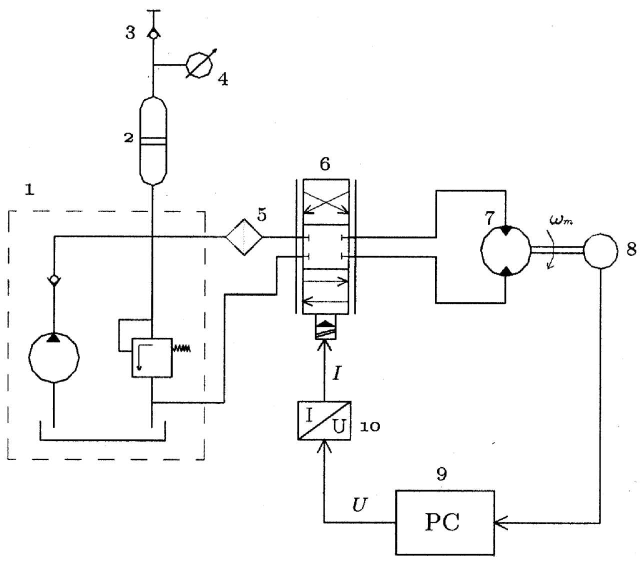


Figure 1: Electrohydraulic velocity servosystem.

a) Determine equilibrium points of this system for $x_1 = x_{1e} = 1$.

b) Using the Lyapunov indirect method study stability properties of equilibrium points determined in a).

b) Check controllability of the linearized system.

c) Check observability in the following two cases:

- when the measured output is the motor angular velocity,
- when the measured output is the load pressure differential.

d) Based on the test that you performed in part c) above, if you could only measure either motor angular velocity or load pressure differential, which one would you choose? Why?

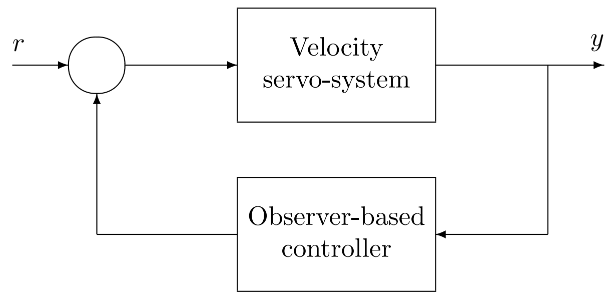
Problem 2: For a system from **Problem 1** design an observer based controller when the measured output is motor angular velocity. You should use LQR design to find K and L to stabilize matrices $A - BK$ and $A - LC$. Experiment with different choices of matrices Q and R.

Problem 3: As a check for your design in **Problem 2**, do the following:

a) Find the transfer function of the linearized system from voltage control input to motor shaft output. You can use the MATLAB commands `ss2tf` or `tfdata` for this. Note the degree of this transfer function.

b) Find the transfer function of the observer based controller. Note the degree of this transfer function.

c) Connect the two transfer functions in feedback as shown in the diagram below, and compute the transfer function from r to y. Note the degree of this transfer function. Is it stable?



Note: you can use the MATLAB command `feedback` to solve this problem. Since this command assumes negative feedback (and here the negative sign is already included in the controller), you will need to multiply the system in the feedback loop with -1 when using this command. If you are unsure how this command works type `help feedback`, and check it first on a simple example for which you can easily determine the answer by hands.

d) Find the poles of the closed-loop transfer function. Check that they are exactly the eigenvalues of $A - BK$ and $A - LC$.

Note: to check stability, you can use the MATLAB function `roots` to find the roots of a polynomial.

Problem 4: Simulate the linearized servo-system in feedback with three different controllers (from initial conditions which are all zeros except $x_1(0) = 0.5$):

a) A state feedback controller using K designed with LQR (use the K designed for the observer based controller).

b) Observer based controller measuring motor angular velocity.

c) Proportional-integral (PI) controller of the form

$$u(t) \;=\; -k_p x_1(t) \;-\; k_i \int_0^t x_1(\tau)\,\mathrm{d}\tau.$$

For what values of gains k_p and k_i is the linearized system stable?

Produce three separate graphs, one for motor angular velocity, one for pressure, and one for control signal. On each of the graphs plot and compare the responses of the three different controllers. Discuss your observations.

Problem 5: Consider the heat equation in a one dimensional medium

$$\frac{\partial \psi}{\partial t}(\xi, t) \;=\; \frac{\partial^2 \psi}{\partial \xi^2}(\xi, t), \quad \xi \in [0,1], \quad t \geq 0,$$

where $\psi(\xi, t)$ is the temperature distribution. We assume that at one end the medium is connected to a heat bath with fixed temperature, (say $\psi(0,t) = 0$), and that at the other end, the temperature can be directly controlled, i.e. $\psi(1,t) = u(t)$, where u is the control input.

The above is an example of so-called infinite dimensional system (dimension of the state-space *infinite*, cannot be represented by rational transfer functions, ...), which can be approximated by a finite dimensional system (but of high dimension!) using for example a finite difference approximation to $\frac{\partial^2}{\partial \xi^2}$. Define the variables

$$\psi_i(t) \;:=\; \psi(i\Delta, t), \quad \Delta = 1/N, \quad i = 0, 1, \ldots, N.$$

The approximate dynamics are then obtained using a second order finite difference approximation to the spatial derivatives, which yields

$$\dot{\psi}_i(t) \;=\; \frac{1}{\Delta^2}\left(\psi_{i+1}(t) - 2\psi_i(t) + \psi_{i-1}(t)\right), \quad i = 1, \ldots, N-1.$$

Note that $\psi_o(t)$ and $\psi_N(t)$ are determined by the boundary conditions.
For the items below, use a fine grid, say $N \geq 50$.

1. Set up this problem in our standard state space form, and indicate what special structure the matrices A and B have.
   Simulate the response of the system (using MATLAB) to some initial nonzero temperature distribution (be creative, try a concentrated initial temperature profile or concentrations in two different locations). Plot the resulting response.
   MATLAB: *To simulate the response use the command* `initial`*, it can provide the response as a matrix (one coordinate is time, the other is state index)*
   To form the required state space matrices use the matlab commands `toeplitz`*.*
   To plot any responses or matrices, use the command `mesh` *which will do a 3D plot of a matrix.*

2. To specify an output, consider two cases:

   (a) The entire temperature profile is the output, i.e. the C matrix is the identity.

   (b) The temperature at the midpoint (say $\psi_{N/2}$ - with N even) is the output.

   For each of those two cases compute the Hankel singular values of these two systems and plot them (use a semilogy plot to fully appreciate this, and note that double precision is roughly at 10^{-16}). To what can you attribute the different decay rates of the singular values?

   Note: the Hankel singular values are the positive square roots of the eigenvalues of PQ, where P is controllability Gramian and Q is observability Gramian. You can use MATLAB command hksv to compute the Hankel singular values.

3. For single output case, use balanced truncation (*use the command schbal*) to find a low order approximation of dimension 2 (i.e., with only 2 states). Compute the responses of the unreduced and reduced system to a unit step input and compare them on the same graph.
   Plot and compare the frequency responses of both the reduced and unreduced systems.

   Note: A balance realization of a stable system is a minimal realization in which controllability and observability Gramians are equal (to each other) and diagonal. In balanced realization each state is equally controllable as it is observable, and the associated Hankel singular value is the appropriate measure of the state's joint controllability and observability. Balanced truncation is obtained by discarding the states with small Hankel singular values, and it represents an efficient way of obtaining reduced order models of stable LTI systems with modest number of state variables (less than 1000).