Due Friday 09/15/06

1. **Reading Assignment:** Chapters 4–8 of Brogan contain a very good coverage of linear algebra. Refer to these chapters throughout the semester. For now, start by reading chapters 4 and 5 to follow the lectures, and pick up what you need to do the homework.

2. Let $\mathbb{H}$ be a linear vector space on which we have the *inner product* $\langle \cdot, \cdot \rangle : \mathbb{H} \times \mathbb{H} \mapsto \mathbb{C}$ satisfying the following axioms:

   - $\langle u, v \rangle = \overline{\langle v, u \rangle}$, where the bar denotes complex conjugation.
   - $\langle u, \alpha_1 v_1 + \alpha_2 v_2 \rangle = \alpha_1 \langle u, v_1 \rangle + \alpha_2 \langle u, v_2 \rangle$, for any $\alpha_1,\ \alpha_2 \in \mathbb{C}$.
   - $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0$ if and only if $v = 0$.

   For $v \in \mathbb{H}$, let $\|v\| = (\langle v, v \rangle)^{\frac{1}{2}}$ denote the *norm* of v. Verify that $\langle \cdot, \cdot \rangle$ satisfies the Cauchy-Schwarz inequality and that $\|\cdot\|$ satisfies the triangle inequality, i.e. $|\langle u, v \rangle| \leq \|u\|\|v\|$, and $\|u+v\| \leq \|u\| + \|v\|$, respectively, for all $u, v \in \mathbb{H}$.

3. Consider the three vectors $x_1 = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$, $x_2 = \begin{bmatrix} 1 & -2 & 3 \end{bmatrix}^T$, $x_3 = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T$.

   (a) Show that this set is linearly independent.

   (b) Generate an orthonormal set using the Gram-Schmidt procedure.

   (c) Express the vector $z = \begin{bmatrix} 6 & 4 & -3 \end{bmatrix}^T$ in terms of the orthonormal basis set obtained above.

4. Let $\mathbb{H}$ be a linear vector space consisting of all 2×2 matrices defined over the real field. Let A, B be any two elements of $\mathbb{H}$.

   (a) Show that $\langle A, B \rangle = \operatorname{trace}(A^T B)$ is a valid inner product (you can actually prove this for $n \times n$ matrices with real entries).

   (b) Show that a basis for the linear space $\mathbb{H}$ is the set of matrices

   $$\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \quad \begin{bmatrix} -2 & -1 \\ 4 & 2 \end{bmatrix}, \quad \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}.$$

   Is this an orthogonal set?

   (c) Obtain a representation of the 2×2 identity matrix in terms of the basis given above.

5. Let $\mathcal{A} : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator. Consider the two sets $B = \{b_1, b_2, b_3\}$ and $C = \{c_1, c_2, c_3\}$ below

   $$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad C = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

   It should be clear to you that these are bases for $\mathbb{R}^3$.

   (a) Find the transformation P relating the two bases.

   (b) Suppose the linear operator $\mathcal{A}$ maps

   $$\mathcal{A} b_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \quad \mathcal{A} b_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathcal{A} b_3 = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}.$$

   Write down the matrix representation of $\mathcal{A}$ with respect to the basis B and also with respect to the basis C.

   Be careful with this one! You know that a similarity transformation is involved – be sure to carefully justify your choice for the required matrix.

6. In Figure 1 a mass M slides over a frictionless wire and is connected to a spring with spring constant k. The spring is in an unstretched position when it is at an angle of 45 degrees from the vertical. The mass is subject to a force input u. The output of the system is the mass displacement z. Obtain a state space description for this system.

   Is this system

   (a) causal,
   (b) time-varying,
   (c) linear,
   (d) memoryless,
   (e) finite-dimensional?

   Explain.

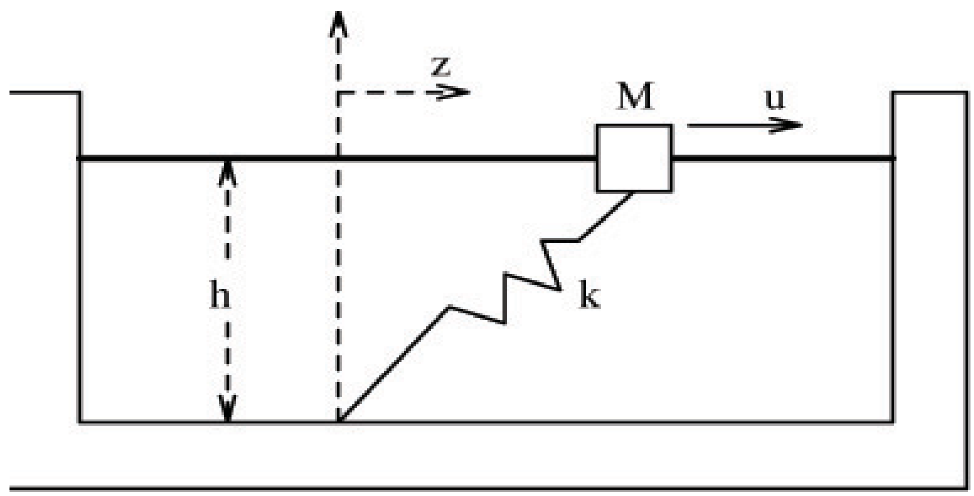

Figure 1: Mass-spring system.