Due Monday 09/22/08

1. In Figure 1 a mass M slides over a frictionless wire and is connected to a spring with spring constant k. The spring is in an unstretched position when it is at an angle of 45 degrees from the vertical. The mass is subject to a force input u. The output of the system is the mass displacement z. Obtain a state space description for this system.

   Is this system

   (a) causal,
   (b) time-varying,
   (c) linear,
   (d) memoryless,
   (e) finite-dimensional?

   Explain.

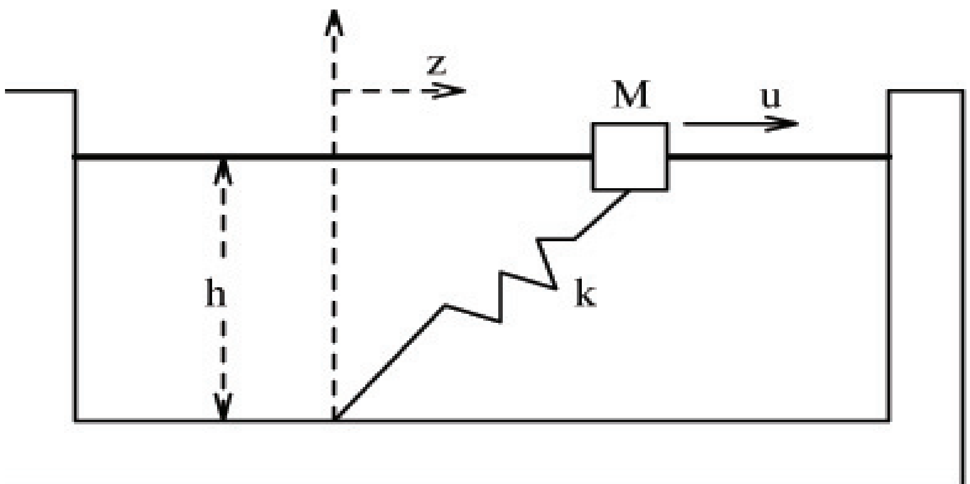

Figure 1: Mass-spring system.

2. Using the state space model that you derived for mass-spring system in Problem 1:

   (a) Find *all* equilibrium points of the system.
   (b) Linearize the system around these equilibrium points.

3. The system shown in Figure 2 is composed of a first order system followed by a saturation element. Which of the following properties does this system have a) causality, b) linearity c) time-invariance? Is the system memoryless? Compute the output y that corresponds to the periodic input in Figure 2 and zero initial conditions.

   Note: The Saturation function works as follows: if the two signals g and y are related by $y(t) = \text{Saturation}\,(g(t))$, then

$$y(t) = \begin{cases} g(t) & \text{if} \quad |g(t)| \le 1 \\ 1 & \text{if} \quad g(t) > 1 \\ -1 & \text{if} \quad g(t) < -1 \end{cases}$$

4. Given the *periodically varying* system $x(k+1) = A(k)x(k) + B(k)u(k)$, where $A(k+N) = A(k)$ and $B(k+N) = B(k)$, define the *sampled state* $z(k)$ and the associated *extended input vector* $v(k)$ by

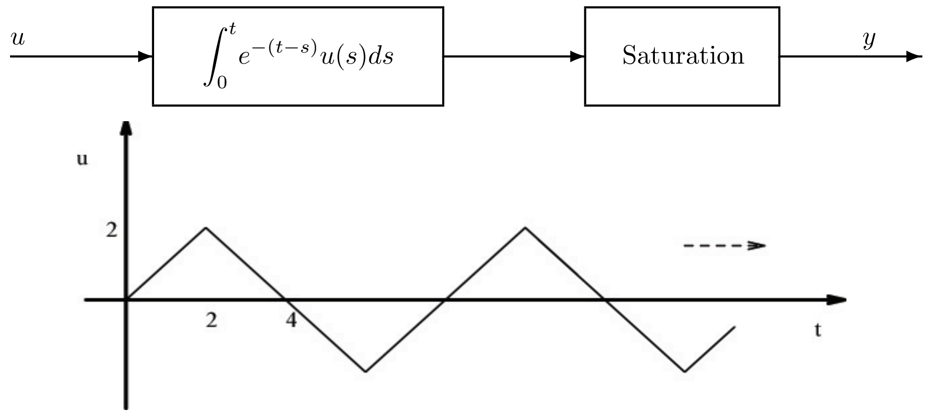

Figure 2: System in Problem 3.

$$z(k) = x(kN) \ , \qquad v(k) = \begin{pmatrix} u(kN) \\ u(kN+1) \\ \vdots \\ u(kN+N-1) \end{pmatrix} .$$

Show that $z(k+1) = Fz(k) + Gv(k)$ for *constant* matrices F and G (i.e. matrices independent of k) by determining F and G explicitly. The above procedure is known as *lifting* and it illustrates how a discrete time-periodic linear system can be transformed to a discrete time-invariant linear system by extending a dimension of its input space.

5. Suppose that we are given a time-invariant system $\dot{x} = Ax + Bu$, $y = Cx$. In order to interface with a digital computer used to compute a control input, we may assume that the input is piecewise constant, say

$$u(t) = u_k \ , \quad k\delta < t \leq (k+1)\delta$$

(where $\{u_k\}$ is a discrete-time signal) and the output $y(\cdot)$ is observed only at the instants $k\delta$, $k = 0, 1, ...$, where δ is some fixed sampling interval. Show that for this purpose, the above continuous-time realization may be replaced by the discrete-time system

$$\mathbf{x}_{k+1} = \Phi \mathbf{x}_k + \Gamma \mathbf{u}_k \ , \quad \mathbf{y}_k = C\mathbf{x}_k$$

where $\mathbf{y}_k := y(k\delta)$, $\mathbf{x}_k := x(k\delta)$, $\Phi := \exp(A\delta)$, $\Gamma := \left(\int_0^\delta [\exp(A\tau)]d\tau\right) B$.

6. (a) Suppose that A and B are constant square matrices. Show that the state transition matrix for the time-varying system described by

$$\dot{x}(t) = e^{-At}Be^{At}x(t)$$

is

$$\Phi(t, s) = e^{-At}e^{(A+B)(t-s)}e^{As} .$$

(b) If A is an $n \times n$ matrix of full rank, show using the definition of the matrix exponential that

$$\int_0^t e^{A\sigma}d\sigma = [e^{At} - I]A^{-1} .$$

Using this result, obtain the solution to the linear time-invariant equation

$$\dot{x} = Ax + B\bar{u} \ , \quad x(0) = x_0 \ ,$$

where $\bar{u}$ is a constant r-dimensional vector and B is an $(n \times r)$-dimensional matrix.