5° M.M of our system: [redacted]

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{K}{M}\cdot\left\{\sqrt{h^2+x_1^2} - h\sqrt{2}\right\}\cdot\frac{x_1}{\sqrt{h^2+x_1^2}} \end{aligned}\right\} \quad \ldots (1)$$

In the HW #2, problem 1, I have found the equilibrium points of the system (1):

$$\underline{x}_1 = \begin{pmatrix} -h \\ 0 \end{pmatrix};\ \underline{x}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix};\ \underline{x}_3 = \begin{pmatrix} h \\ 0 \end{pmatrix}$$

and linearization has been done:

1° $$A_1 = A_3 = \begin{pmatrix} 0 & 1 \\ -\frac{K}{2M} & 0 \end{pmatrix}$$ linearization around $\underline{x}_1 = \begin{pmatrix} -h \\ 0 \end{pmatrix}$ & $\underline{x}_3 = \begin{pmatrix} h \\ 0 \end{pmatrix}$

2° $$A_2 = \begin{pmatrix} 0 & 1 \\ \frac{(\sqrt{2}-1)K}{M} & 0 \end{pmatrix}$$

Let's try to investigate the stability properties of $\underline{x}_1, \underline{x}_2$ & $\underline{x}_3$ by the use of Lyapunov's Indirect Method

1° $$(sI - A_1) = (sI - A_3) = \begin{pmatrix} s & -1 \\ \frac{K}{2M} & s \end{pmatrix} \Rightarrow$$

$$\Rightarrow \det(sI - A_1) = \det(sI - A_3) = s^2 + \frac{K}{2M} \Rightarrow$$

$$\Rightarrow s_{1,2} = \pm j\sqrt{\frac{K}{2M}}$$ $\Rightarrow$ We cannot make any conclusion about stability of $\underline{x}_1$ & $\underline{x}_3$ using L.I.M.

2° $$(sI - A_2) = \begin{pmatrix} s & -1 \\ -\frac{(\sqrt{2}-1)K}{M} & s \end{pmatrix} \Rightarrow \det(sI - A_2) = s^2 - \frac{(\sqrt{2}-1)K}{M}$$

$\Rightarrow s_1 = \sqrt{\frac{(\sqrt{2}-1)K}{M}}$; $s_2 = -\sqrt{\frac{(\sqrt{2}-1)K}{M}} \Rightarrow$

$\Rightarrow$ there exists eigenvalue in RHP $\Rightarrow$

$\Rightarrow \underline{x} = 0_x$ of the system (1) is not stable equilibrium point.

Let's investigate properties of the equilibrium point $\underline{x}_3 = \begin{pmatrix} h \\ 0 \end{pmatrix}$ by the use of the Lyapunov Direct Method. In order to do this we need to shift coordinate system:

$$\left.\begin{aligned} x_1 &= \bar{x}_1 + h \\ x_2 &= \bar{x}_2 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} \dot{x}_1 &= \dot{\bar{x}}_1 \\ \dot{x}_2 &= \dot{\bar{x}}_2 \end{aligned}\right\} \xrightarrow{(1)}$$

$$\left.\begin{aligned} \dot{\bar{x}}_1 &= \bar{x}_2 \\ \dot{\bar{x}}_2 &= -\frac{K}{M}\cdot\left\{\sqrt{h^2+(h+\bar{x}_1)^2} - h\sqrt{2}\right\}\cdot\frac{(\bar{x}_1+h)}{\sqrt{h^2+(\bar{x}_1+h)^2}} \end{aligned}\right\} \Rightarrow (2)$$

$$\Rightarrow \left.\begin{aligned} x_1 = h &\Rightarrow \bar{x}_1 = 0 \\ x_2 = 0 &\Rightarrow \bar{x}_2 = 0 \end{aligned}\right\}$$

Adopt function $V(\bar{x})$ in the form:

$$V(\bar{x}) = \frac{2K}{M}\cdot\left\{\frac{3h^2}{2} + \frac{(\bar{x}_1+h)^2}{2} - h\sqrt{2}\sqrt{h^2+(h+\bar{x}_1)^2}\right\} + \bar{x}_2^2$$

Properties of the function $V(\bar{x})$:

1° $V(\bar{x}) \in C(\mathbb{R}^2)$, $\forall \bar{x} \in \mathbb{R}^2$

2° $\boxed{V(0_x)} = \frac{2K}{M}\cdot\left\{\frac{3h^2}{2} + \frac{h^2}{2} - h\sqrt{2}\cdot\sqrt{h^2+h^2}\right\} + 0 =$

$= \frac{2K}{M}\cdot\left\{2h^2 - h\sqrt{2}\cdot h\sqrt{2}\right\} = \frac{2K}{M}\cdot\left\{2h^2 - 2h^2\right\} = \boxed{0}$

3° Claim: $V(\overline{x}) > 0$, $\forall \overline{x} \in \mathcal{Y} \setminus \{0_x\}$ $\mathcal{Y} = \{\underline{x};\ x_1 \geq -2h \wedge x_2 \in \mathbb{R}$

For the sake of simplicity let's write x_1 and x_2 instead of $\overline{x}_1$ and $\overline{x}_2$.

$$\frac{3h^2}{2} + \frac{(x_1+h)^2}{2} - \sqrt{2}\,h\sqrt{(x_1+h)^2+h^2} = \left[\sqrt{\frac{(x_1+h)^2+h^2}{2}} - h\right]^2$$

Claim:

$$\frac{3h^2}{2} + \frac{(x_1+h)^2}{2} > h\sqrt{2}\cdot\sqrt{h^2+(h+x_1)^2}, \quad \forall x_1 \in \mathbb{R} \setminus \{\{0\} \cup \{-2h\}\}$$

Proof: Let's introduce replacement: $z = (x_1+h)^2 \Rightarrow$

$$\Rightarrow \frac{1}{2}\cdot(3h^2+z) > h\sqrt{2}\sqrt{h^2+z} \quad \Big|^2$$

(both sides are positive $\Rightarrow$ $a > b \,|^2 \Leftrightarrow a^2 > b^2$ for $a>0, b>0$)

$$\frac{1}{4}\cdot(3h^2+z)^2 > 2h^2(h^2+z) \quad \Big| \cdot 4$$

$$\Rightarrow 9h^4 + 6h^2z + z^2 > 8h^2(h^2+z) \Rightarrow$$

$$\Rightarrow 9h^4 + 6h^2z + z^2 > 8h^4 + 8h^2z \Rightarrow$$

$$\Rightarrow h^4 - 2h^2z + z^2 > 0 \Rightarrow$$

$$\Rightarrow (h^2 - z)^2 > 0 \Rightarrow (h^2 - (h+x_1)^2)^2 > 0$$

$$\Rightarrow \{(h-h-x_1)(h+h+x_1)\}^2 > 0 \Rightarrow$$

$$\Rightarrow \{-x_1\cdot(2h+x_1)\}^2 > 0 \Rightarrow x_1^2(2h+x_1)^2 > 0$$

This can be equal to 0 just in the case that $x_1 = 0 \vee x_1 = -2h \Rightarrow$

$\Rightarrow V(\underline{x})$ is positive definite function on the set $\mathcal{Y} = \{\underline{x} \in \mathbb{R}^2;\ x_1 > -2h \wedge x_2 \in \mathbb{R}\}$

$$\dot{V}(\underline{x}) = \frac{\partial V}{\partial x_1}\cdot \dot{x}_1 + \frac{\partial V}{\partial x_2}\cdot \dot{x}_2$$

$$\boxed{\frac{\partial V}{\partial x_1}} = \frac{2K}{M}\cdot\frac{\partial}{\partial x_1}\left\{\frac{3h^2}{2} + \frac{(x_1+h)^2}{2} - h\sqrt{2}\sqrt{h^2+(h+x_1)^2}\right\} =$$

$$= \frac{2K}{M}\left\{\frac{1}{\cancel{2}}\cdot \cancel{2}(x_1+h) - h\sqrt{2}\,\frac{\cancel{2}(h+x_1)}{\cancel{2}\sqrt{h^2+(h+x_1)^2}}\right\} =$$

$$= \frac{2K}{M}\cdot\left\{(x_1+h) - h\sqrt{2}\,\frac{(h+x_1)}{\sqrt{h^2+(h+x_1)^2}}\right\} =$$

$$\boxed{= \frac{2K}{M}\cdot\left\{1 - \frac{h\sqrt{2}}{\sqrt{h^2+(h+x_1)^2}}\right\}(h+x_1)}$$

$$\boxed{\frac{\partial V}{\partial x_2} = 2x_2} \Rightarrow$$

$$\Rightarrow \boxed{\dot{V}(\underline{x})} = \frac{2K}{M}\cdot\left(1 - \frac{h\sqrt{2}}{\sqrt{h^2+(h+x_1)^2}}\right)(h+x_1)x_2 +$$

$$+ 2x_2\cdot\left(-\frac{K}{M}\cdot\left\{1 - \frac{h\sqrt{2}}{\sqrt{h^2+(h+x_1)^2}}\right\}(x_1+h)\right) = \boxed{0} \Rightarrow$$

$\Rightarrow V(\underline{x})$ is Lyapunov function of the system $(2) \Rightarrow$ equilibrium point $\underline{x} = \underline{0}_x$ of the system (2) is stable in the sense of Lyapunov $\Rightarrow$

$\Rightarrow$ E.P. $\underline{x} = \begin{pmatrix} h \\ 0 \end{pmatrix}$ of the system (1) is stable in the sense of Lyapunov.

1. Investigating the stability properties of the e.p.

$$V(\underline{x}) = \frac{2K}{M}\left\{\frac{3h^2}{2} + \frac{1}{2}(x_1 - h)^2 - h\sqrt{2}\sqrt{h^2 + (x_1 - h)^2}\right\} + x_2^2$$

Properties of $V(\underline{x})$:

1° $V(\underline{0}_x) = \frac{C(\mathbb{R}^2)}{M}\{\frac{1}{2} + \frac{x}{2}$, $\forall \underline{x} \in \mathbb{R}^2$ $\}$ $M)$

$\Rightarrow V(\underline{0}_x) = 0$

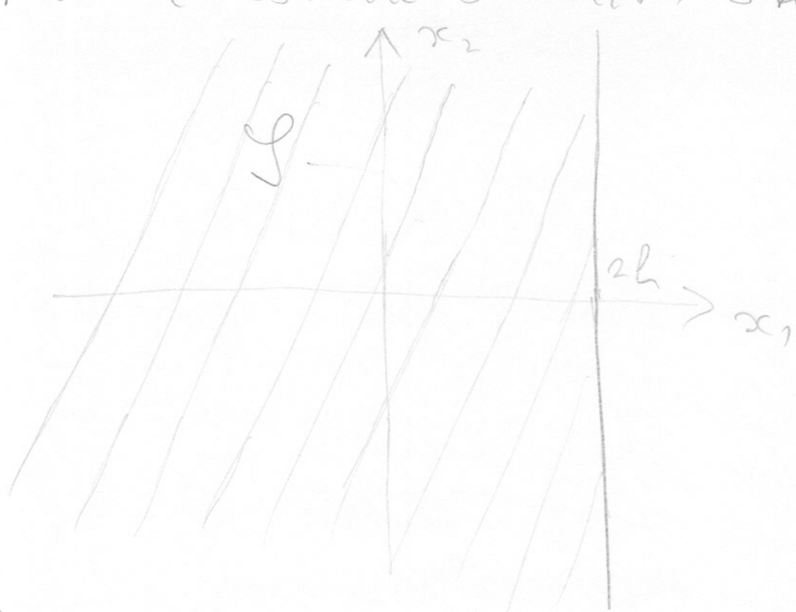
3° Claim $V(\underline{x}) > 0$, $\forall \underline{x} \in \mathcal{S} \setminus \{\underline{0}_x\}$; $\mathcal{S} = \{\underline{x} \in \mathbb{R}^2;\ x_1 < 2h \wedge x_2 \in \mathbb{R}\}$

Pf: $(x_1 - h)^2 = z$ In the similar way as above one can

$\Leftrightarrow (2h - x_1)^2 \cdot x_1^2 > 0$ – This is true for $\forall x_1 \in \mathbb{R} \setminus \{x_1 = 0 \vee x_1 = 2h\}$

5-5

$\Rightarrow v(\underline{x})$ is positive definite function on $S = \{\underline{x} \in \mathbb{R}^2;\ x_1 < 2h;\ x_2 \in \mathbb{R}\}$



$$\dot{v}(\underline{x}) = \frac{\partial v}{\partial x_1}\cdot \dot{x}_1 + \frac{\partial v}{\partial x_2}\cdot \dot{x}_2$$

$$\frac{\partial v}{\partial x_2} = 2x_2$$

$$\boxed{\frac{\partial v}{\partial x_1}} = \frac{2K}{M}\cdot\left\{\frac{1}{2}\cdot 2(x_1-h) - h\sqrt{2}\,\frac{2(x_1-h)}{2\sqrt{h^2+(x_1-h)^2}}\right\} =$$

$$= \boxed{\frac{2K}{M}\left\{1 - \frac{h\sqrt{2}}{\sqrt{h^2+(x_1-h)^2}}\right\}\cdot(x_1-h)} \Rightarrow$$

$$\Rightarrow \boxed{\dot{v}(\underline{x})} = \frac{2K}{M}\cdot\left\{1 - \frac{h\sqrt{2}}{\sqrt{h^2+(x_1-h)^2}}\right\}(x_1-h)x_2 +$$

$$+ 2x_2\cdot\left\{-\frac{K}{M}\cdot\left(1 - \frac{h\sqrt{2}}{\sqrt{h^2+(x_1-h)^2}}\right)(x_1-h)\right\} = \boxed{0}$$

$\Rightarrow v(\underline{x})$ is the Lyapunov function of the system (3) $\Rightarrow$
$\Rightarrow$ E.P. $\underline{x} = \underline{0}_x$ of the system (3) is stable in the sense of Lyapunov $\Rightarrow$ E.P. $\underline{x} = \begin{pmatrix} -h \\ 0 \end{pmatrix}$ of the system (1) is stable in the sense of Lyapunov. ∎

6° $\dot{x}(t) = A(t)\,x(t) \quad \dots (1)$

$A(t+T) = A(t) \quad \dots (2)$

[redacted]

a) $\left.\begin{array}{l} \dot{\Phi}(t,0) = A(t)\,\Phi(t,0) \\ \Phi(0,0) = I \end{array}\right\} \quad \begin{array}{l} \dots (3) \\ \dots (4) \end{array}$

Show that $\Psi(t,0) = \Phi(t+T,0)$ $\dots (5)$ satisfies:

$$\dot{\Psi}(t,0) = A(t)\cdot\Psi(t,0) \quad \dots (6)$$

$$\Psi(0,0) = \Phi(T,0) \quad \dots (7)$$

$\boxed{\Psi(0,0)} = \Phi(0+T,0) = \boxed{\Phi(T,0)} \Rightarrow (5)$ satisfies (7)

$\boxed{\dot{\Psi}(t,0)} = \dot{\Phi}(t+T,0) \overset{(3)}{=} A(t+T)\cdot\Phi(t+T,0) \overset{(2),(5)}{=\!=}$

$= \boxed{A(t)\cdot\Psi(t,0)} \Rightarrow (5)$ satisfies (6)

b) Show that

$$\Phi(t+T,0) = \Phi(t,0)\cdot\Phi(T,0) \quad \dots (8)$$

is true.

$[(5),(7)] \rightarrow (3) \Rightarrow \Psi(t,0) = \Phi(t,0)\cdot\Psi(0,0) \quad \dots (9)$

(9) must satisfy (6) and (7)

$(9) \rightarrow \boxed{\Psi(0,0)} = \Phi(0,0)\cdot\Psi(0,0) \overset{(4)}{=} I\cdot\Psi(0,0) = \boxed{\Psi(0,0)}$ (9) satisfie

$\boxed{\dot{\Psi}(t,0)} = \dot{\Phi}(t,0)\cdot\Psi(0,0) \overset{(3)}{=} A(t)\cdot\Phi(t,0)\cdot\Psi(0,0) \overset{(9)}{=} \boxed{A(t)\cdot\Psi(t,0}$

$\Rightarrow$ (9) satisfies (6)

c) $\exists\, \Phi^{-1}(T,0) \Rightarrow \Phi(T,0) = e^{TR}$?

Jacobi-Liouville formula $\longrightarrow$

$$\det\{\Phi(t,t_0)\} = \exp\left\{\int_{t_0}^{t} \mathrm{trace}[A(\tau)]\,d\tau\right\} \Rightarrow$$

$$\Rightarrow \det\{\Phi(T,0)\} = \exp\left\{\int_{0}^{T} \mathrm{trace}[A(\tau)]\,d\tau\right\}$$

IF

$$A(t) = \begin{bmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & & \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{bmatrix}$$

$$\mathrm{trace}[A(t)] = \sum_{i=1}^{n} a_{ii}(t) = f(t)$$

$$a_{ij}(t) = a_{ij}(t+T),\ \forall (i,j)=1,2,\ldots,n,\ \forall\, t \geq t_0 \Rightarrow$$

$$\Rightarrow a_{ij}(0) = a_{ij}(T),\ \forall (i,j) = 1,2,\ldots,n \Rightarrow$$

$$\Rightarrow f(0) = f(T) \Rightarrow$$ [redacted]

$$\int_0^T f(\tau)\,d\tau = g(\tau)\Big|_0^T = g(T) - g(0) = 0 \Rightarrow$$

$$\Rightarrow \boxed{\det\{\Phi(T,0)\}} = e^0 = \boxed{1} \Rightarrow$$

$$\Rightarrow \exists\ \Phi^{-1}(T,0) \Rightarrow$$

$$\Phi(T,0) = e^{TR},\ R \in \mathbb{R}^{n\times n} \quad \ldots (10) \Rightarrow$$

$$\Rightarrow \Phi^{-1}(T,0) = e^{-TR} \quad \ldots (11)$$

---

d) $P(t)^{-1} = \Phi(t,0)\cdot e^{-tR} \quad \ldots (12)$

1° Show that $P(t)^{-1}$ and $P(t)$ are periodic with period T.

2° Show that $P(T) = I$

2° $\boxed{P(T)^{-1}} = \Phi(T,0)\cdot e^{-TR} \overset{(10)}{=} e^{TR}\cdot e^{-TR} = \boxed{I} \Rightarrow P(T) = I$

1° $\boxed{P(t+T)^{-1}} = \Phi(t+T,0)\cdot e^{-(t+T)R} \overset{(8)}{=} \Phi(t,0)\,\Phi(T,0)\cdot e^{-(t+T)R} =$

$\overset{(10)}{=} \Phi(t,0)\cdot e^{TR}\cdot e^{-(t+T)R} = \Phi(t,0)\cdot e^{(T-t-T)\cdot R} = \Phi(t,0)\cdot e^{-t\cdot R} \overset{(12)}{=} \boxed{P(t)^{-1}} \Longrightarrow$

$P(t)^{-1}$, $P(t)$ are periodic with period T.

$$\Phi(t,0) = P(t)^{-1} \cdot e^{tR} \quad \ldots (13)$$

$$\Phi(0,t_0) \cdot \Phi(t_0,0) = I \quad \Big| \cdot \Phi^{-1}(t_0,0)$$

$$\boxed{\Phi(0,t_0) = \Phi^{-1}(t_0,0)} \quad \ldots (14)$$

2° $\boxed{\Phi(t,t_0)} = \Phi(t,0) \cdot \Phi(0,t_0) \overset{(13)}{=}$

$= P(t)^{-1} \cdot e^{tR} \cdot \Phi(0,t_0) \overset{(14)}{=} P(t)^{-1} e^{tR} \cdot \Phi^{-1}(t_0,0) \overset{(13)}{=}$

$= P(t)^{-1} \cdot e^{tR} \cdot \left\{ P(t_0)^{-1} \cdot e^{t_0 R} \right\}^{-1} =$

$= P(t)^{-1} \cdot e^{tR} \cdot e^{-t_0 R} \cdot P(t_0) =$

$$= \boxed{P(t)^{-1} \cdot e^{(t-t_0)R} \cdot P(t_0)}$$

Problem 8:

$$\dot{x}_1 = -x_1^2 + x_1 x_2 \quad \text{--- (1)}$$

$$\dot{x}_2 = -2x_2^2 + x_2 - x_1 x_2 + 2 \quad \text{--- (2)}$$

Solution: (a) plug in $\bar{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, we have

$\dot{x}_1 = 0$

$\dot{x}_2 = -2+1-1+2 = 0$. $\bar{x}$ is an E.P.

(b) It is easy to verify that $\bar{\bar{x}} = \begin{pmatrix} -\frac{2}{3} \\ -\frac{2}{3} \end{pmatrix}$ is another E.P., thus $\bar{x}$ is not the only E.P.

(c) Linearized model at $\bar{x}$

$$\cancel{\dot{x} = \begin{pmatrix} -2x_1+x_2 & -4x_2+1-x_1 \\ x_1 & \end{pmatrix}}$$

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}\Bigg|_{\bar{x}} = \begin{pmatrix} -2x_1+x_2 & x_1 \\ -x_2 & -4x_2+1-x_1 \end{pmatrix}\Bigg|_{\bar{x}}$$

$$= \begin{pmatrix} -1 & 1 \\ -1 & -4 \end{pmatrix}$$

trace$(A) = -5 < 0$, det$(A) = 5 > 0$ $\Rightarrow$ $\lambda_1, \lambda_2 \in$ L.H.P. (actually, $\lambda_1 = -1.3820$, $\lambda_2 = -3.6180$)

thus, the system is A.S. at $\bar{x}$.

(d) No; since there is more than one E.P.

$3°$ $$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 - x_1^3 - \delta x_2, \; \delta > 0 \end{aligned}\right\} \cdots (1) \iff \dot{\underline{x}} = \underline{f}(\underline{x})$$

[redacted]

a) E.P.
$$x_2 = 0 \qquad \cdots (2)$$
$$x_1 - x_1^3 - \delta x_2 = 0 \qquad \cdots (3)$$

$(2) \to (3) \Rightarrow x_1(1 - x_1^2) = 0 \to x_1 = 0 \vee x_1 = 1 \vee x_1 = -1 \Rightarrow$

$\Rightarrow$ Candidates for E.P. are

$$\underline{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}; \; \underline{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \; \underline{x} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

Since $\underline{f}(\underline{x}) \in C^{(1)}(\mathbb{R} \times \mathbb{R}) \Rightarrow$ Equilibrium points of the system (1) are:

$$\underline{x}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}; \; \underline{x}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \; \underline{x}_3 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

b) Let's use Lyapunov's Indirect Method for determining stability properties of $\underline{x}_1$, $\underline{x}_2$ and $\underline{x}_3$:

$$A_i = \left.\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}\right|_{\underline{x} = \underline{x}_i}, \quad i = 1, 2, 3$$

$$\left.\begin{aligned} \frac{\partial f_1}{\partial x_1} &= 0; \; \frac{\partial f_1}{\partial x_2} = 1 \\ \frac{\partial f_2}{\partial x_1} &= 1 - 3x_1^2; \; \frac{\partial f_2}{\partial x_2} = -\delta \end{aligned}\right\} \Rightarrow \begin{aligned} \frac{\partial f_1}{\partial x_1} &= \text{const}; \; \frac{\partial f_1}{\partial x_2} = \text{const} \\ \frac{\partial f_2}{\partial x_2} &= \text{const}; \; \frac{\partial f_2}{\partial x_1} \neq \text{const} \end{aligned}$$

$$\boxed{\left.\frac{\partial f_2}{\partial x_1}\right|_1} = 1 - 3 \cdot 0 = \boxed{1}$$

$$\boxed{\left.\frac{\partial f_2}{\partial x_1}\right|_2 = \left.\frac{\partial f_2}{\partial x_1}\right|_3} = 1 - 3 \cdot (\pm 1)^2 = 1 - 3 = \boxed{-2}$$

$\longrightarrow$

$$A_1 = \begin{pmatrix} 0 & 1 \\ 1 & -\delta \end{pmatrix}; \quad A_2 = A_3 = \begin{pmatrix} 0 & 1 \\ -2 & -\delta \end{pmatrix}$$

Eigenvalues of matrix A_1:

$$(sI - A_1) = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & -\delta \end{pmatrix} = \begin{pmatrix} s & -1 \\ -1 & s+\delta \end{pmatrix}$$

$$\det(sI - A_1) = s^2 + \delta s - 1$$

Since the characteristic polynom of the matrix A_1 does not have coeficients of the same sign $\Rightarrow$
$\Rightarrow \exists$ eigenvalue of the matrix A_1 with positive real part $\Rightarrow$ matrix A_1 is not a stable matrix $\Rightarrow$ $\underline{x}_1 = \underline{0}_x$ is not stable equilibrium point of the system (1).

Characteristic matrix of matrices A_2 and A_3

$$(sI - A_2) = (sI - A_3) = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -2 & -\delta \end{pmatrix} =$$

$$= \begin{pmatrix} s & -1 \\ 2 & s+\delta \end{pmatrix} \Rightarrow$$

$$\Rightarrow \det(sI - A_2) = \det(sI - A_3) = s^2 + \delta s + 2 \Rightarrow$$

$\Rightarrow$ All eigenvalues of matrices A_2 and A_3 are in the open left hand side plane $\Rightarrow$ $\underline{x}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\underline{x}_3 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ are asymptotically stable equilibrium points of the system (1). ∎

1° Π#1 $u(t)\equiv 0$

1) $x(0)=\begin{pmatrix}1\\-1\end{pmatrix} \Rightarrow x(t)=\begin{pmatrix}1\\-1\end{pmatrix}e^{-t}$; $y(t)=e^{-t}$

2) $x(0)=\begin{pmatrix}1\\1\end{pmatrix} \Rightarrow x(t)=\begin{pmatrix}1\\1\end{pmatrix}e^{t}$

Π#2 $u(t)=1$, $\forall\, t\geq 0$

$x(0)=\begin{pmatrix}1\\0\end{pmatrix} \Rightarrow y(t)\equiv 0$

a) $\left.\begin{matrix}\underline{\Phi}(t)=e^{tA}=?\\ \text{Π\#1}\end{matrix}\right\}$ $\underline{x}(t)=\begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix}=\underline{\Phi}(t)\cdot\underline{x}(0)=\begin{pmatrix}\phi_{11} & \phi_{12}\\ \phi_{21} & \phi_{22}\end{pmatrix}\begin{pmatrix}x_{10}\\x_{20}\end{pmatrix}$

1) $\begin{pmatrix}e^{-t}\\-e^{-t}\end{pmatrix}=\begin{pmatrix}\phi_{11} & \phi_{12}\\ \phi_{21} & \phi_{22}\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix}\Rightarrow$

[redacted]

$\phi_{11}-\phi_{12}=e^{-t} \quad \dots(1)$

$\phi_{21}-\phi_{22}=-e^{-t} \quad \dots(2)$

2) $\begin{pmatrix}e^{t}\\e^{t}\end{pmatrix}=\begin{pmatrix}\phi_{11} & \phi_{12}\\ \phi_{21} & \phi_{22}\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}\Rightarrow$

$\phi_{11}+\phi_{12}=e^{t} \quad \dots(3)$

$\phi_{21}+\phi_{22}=e^{t} \quad \dots(4)$

$(1)+(3) \Rightarrow 2\phi_{11}=e^{-t}+e^{t} \Rightarrow \phi_{11}(t)=\frac{1}{2}(e^{-t}+e^{t})\dots(5)$

$(5)\to(3) \Rightarrow \boxed{\phi_{12}}=e^{t}-\frac{1}{2}(e^{-t}+e^{t})=\boxed{\frac{1}{2}\cdot(e^{t}-e^{-t})}\dots(6)$

$(2)+(4) \Rightarrow 2\phi_{21}=e^{t}-e^{-t} \Rightarrow \phi_{21}(t)=\frac{1}{2}(e^{t}-e^{-t})\dots(7)$

$(7)\to(4)\to \boxed{\phi_{22}(t)}=e^{t}-\frac{1}{2}(e^{t}-e^{-t})=\boxed{\frac{1}{2}\cdot(e^{t}+e^{-t})} \quad \dots(8)$

2° $\left.\begin{aligned}\dot{x}_1 &= (x_1 - x_2)(x_1^2 + x_2^2 - 1) \\ \dot{x}_2 &= (x_1 + x_2)(x_1^2 + x_2^2 - 1)\end{aligned}\right\} \dots (1) \Rightarrow \underline{\dot{x}} = \underline{f}(\underline{x})$

[redacted]

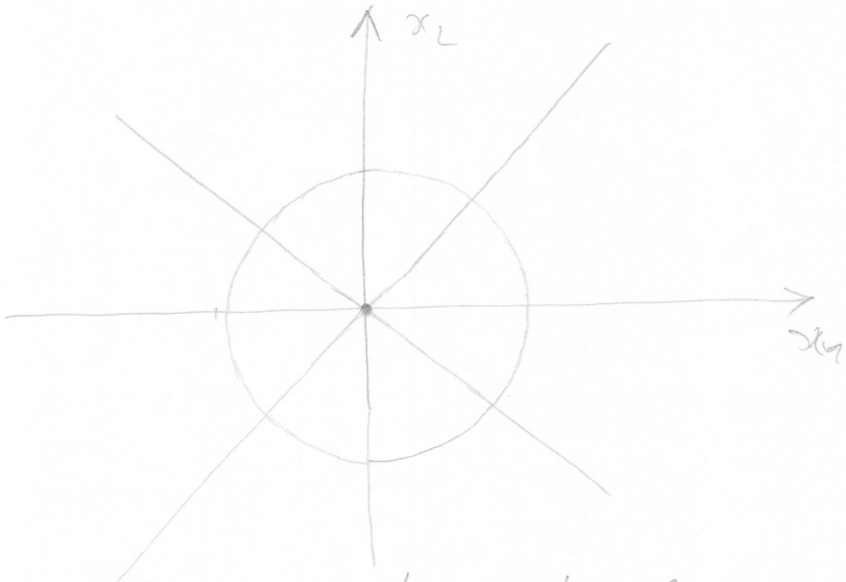
E.P. $(x_1 - x_2)(x_1^2 + x_2^2 - 1) = 0 \quad \dots (2)$

$(x_1 + x_2)(x_1^2 + x_2^2 - 1) = 0 \quad \dots (3)$

$(2) \rightarrow \underbrace{x_1 = x_2}_{(4)} \ \vee \ \underbrace{x_1^2 + x_2^2 = 1}_{(5)}$

$(3) \rightarrow \underbrace{x_1 = -x_2}_{(6)} \ \vee \ \underbrace{x_1^2 + x_2^2 = 1}_{(7)}$

We need to find intersection of the (4), (5), (6) & (7)



$\Rightarrow$

$\left.\begin{aligned}&\text{(2) \& (3) are always satisfied when } x_1^2 + x_2^2 = 1 \\ &\text{They are also satisfied when } x_1 = x_2 = 0\end{aligned}\right\} \Rightarrow$

Candidates for E.P.

$\underline{x} = \underline{0}_x \ \vee \ \mathcal{S} = \{ \underline{x} \in \mathbb{R}^2 ; \ x_1^2 + x_2^2 = 1 \}$

Take function $v(\underline{x})$ in the form:

$v(\underline{x}) = x_1^2 + x_2^2$

Function $v(\underline{x})$ is globally positive definite and radially unbounded (Please see Ex. #1°).

$$\dot{v}(\underline{x}) = \frac{\partial v}{\partial x_1}\cdot\dot{x}_1 + \frac{\partial v}{\partial x_2}\cdot\dot{x}_2 = 2x_1\cdot\dot{x}_1 + 2x_2\cdot\dot{x}_2 =$$

$$= 2x_1\cdot\{(x_1 - x_2)(x_1^2 + x_2^2 - 1)\} + 2x_2\{(x_1 + x_2)(x_1^2 + x_2^2 - 1)\} =$$

$$= 2(x_1^2 + x_2^2 - 1)\cdot\{x_1^2 - \cancel{x_1x_2} + \cancel{x_1x_2} + x_2^2\} =$$

$$= -2(x_1^2 + x_2^2)(1 - x_1^2 - x_2^2) < 0,$$

$\forall\, \underline{x} \in S_1 \setminus \{\underline{0}_x\},$

where $S_1 = \{\underline{x} \in \mathbb{R}^2,\ x_1^2 + x_2^2 < 1\}$

and $\dot{v}(\underline{0}_x) = 0$

$\Rightarrow$ $v(\underline{x})$ is Lyapunov function of the system (1) $\Rightarrow$

$\Rightarrow$ $\underline{x} = \underline{0}_x$ is locally asymptotically stable equilibrium point. ■

(5), (6), (7), (8) $\Rightarrow$

$$\Phi(t) = e^{At} = \frac{1}{2} \cdot \begin{pmatrix} (e^{t} + e^{-t}) & (e^{t} - e^{-t}) \\ (e^{t} - e^{-t}) & (e^{t} + e^{-t}) \end{pmatrix}$$

b) $u(t) \equiv 0$; $\underline{x}(0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$; Since $u(t) \equiv 0 \Rightarrow$

$$\boxed{\underline{x}(t)} = \Phi(t) \cdot \underline{x}(0) = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2\,\phi_{11}(t) \\ 2\,\phi_{21}(t) \end{pmatrix} =$$

$$\boxed{= \begin{pmatrix} (e^{t} + e^{-t}) \\ (e^{t} - e^{-t}) \end{pmatrix}}$$

c) $\dot{\underline{x}}(t) = A\underline{x} + B \cdot u$, where $A \in \mathbb{R}^{2\times 2}$; $B \in \mathbb{R}^{2\times 1}$; $C \in \mathbb{R}^{1\times 2}$

$y(t) = C \cdot \underline{x}$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} ; B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} ; C = (c_1 \quad c_2)$$

$$(sI - A)^{-1} = \mathcal{L}\{\Phi(t)\} = \begin{pmatrix} R_{11}(s) & R_{12}(s) \\ R_{21}(s) & R_{22}(s) \end{pmatrix}$$

$$\boxed{R_{11}(s) = R_{22}(s)} = \mathcal{L}\left\{\frac{1}{2} \cdot (e^{t} + e^{-t})\right\} = \frac{1}{2} \cdot \left\{\mathcal{L}\{e^{t}\} + \mathcal{L}\{e^{-t}\}\right\}$$

$$= \frac{1}{2} \cdot \left\{\frac{1}{s-1} + \frac{1}{s+1}\right\} = \frac{1}{2} \cdot \frac{s+1+s-1}{(s-1)(s+1)} = \boxed{\frac{s}{(s-1)(s+1)}}$$

$$\boxed{R_{12}(s) = R_{21}(s)} = \mathcal{L}\left\{\frac{1}{2}(e^{t} - e^{-t})\right\} = \frac{1}{2} \cdot \left\{\mathcal{L}\{e^{t}\} - \mathcal{L}\{e^{-t}\}\right\} =$$

$$= \frac{1}{2} \cdot \left\{\frac{1}{s-1} - \frac{1}{s+1}\right\} = \frac{1}{2} \cdot \frac{s+1-s+1}{(s-1)(s+1)} = \boxed{\frac{1}{(s-1)(s+1)}} \Rightarrow$$

$$(sI-A)^{-1}=\frac{1}{(s-1)(s+1)}\cdot\begin{pmatrix} s & 1 \\ 1 & s \end{pmatrix}=\frac{1}{\det(sI-A)}\cdot \operatorname{adj}(sI-A) \Rightarrow$$

$$\Rightarrow \det(sI-A)=(s-1)(s+1)$$

$$\operatorname{adj}(sI-A)=\begin{pmatrix} s & 1 \\ 1 & s \end{pmatrix} \Rightarrow sI-A=\begin{pmatrix} s & -1 \\ -1 & s \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} s-a_{11} & -a_{12} \\ -a_{21} & s-a_{22} \end{pmatrix}=\begin{pmatrix} s & -1 \\ -1 & s \end{pmatrix} \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l} a_{11}=0\ ;\ a_{12}=1 \\ a_{21}=1\ ;\ a_{22}=0 \end{array}\right\} \Rightarrow \boxed{A=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$$

T#1 $y(t)=-e^{-t}$, $\underline{x}(0)=\begin{pmatrix} 1 \\ -1 \end{pmatrix}$; $\underline{x}(t)=\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-t}$

$$y(t)=(c_1\ \ c_2)\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}=c_1x_1(t)+c_2x_2(t)$$

$$-e^{-t}=c_1e^{-t}-c_2e^{-t} \Rightarrow (c_1-c_2+1)e^{-t}=0 \Rightarrow$$

$$\Rightarrow c_1-c_2+1=0 \Rightarrow \boxed{c_2=c_1+1} \quad \ldots (9)$$

T#2

$$y(t)\equiv 0 \Rightarrow c_1x_1(t)+c_2x_2(t)=0 \Rightarrow$$

$$\Rightarrow c_1x_1(t)+(c_1+1)x_2(t)=0 \Rightarrow$$

$$\Rightarrow \boxed{c_1[x_1(t)+x_2(t)]+x_2(t)=0} \quad \ldots (10)$$

Input and initial condition are given: $\Rightarrow$

$\Rightarrow$ We can determine $\underline{x}(t)$ as a function of k_1, k_2 and t. $\Rightarrow$

$$\boxed{\underline{x}(t)} = \Phi(t)\cdot\underline{x}(0) + \int_0^t \Phi(t-\tau) B\cdot u(\tau)\, d\tau \overset{u(t)\equiv 1}{=}$$

$$= \boxed{\Phi(t)\cdot\underline{x}(0) + \int_0^t \Phi(t-\tau)\cdot B\cdot d\tau} \quad \dots (11)$$

$$\boxed{\Phi(t)\cdot\underline{x}(0)} = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \phi_{11} \\ \phi_{21} \end{pmatrix} = \boxed{\frac{1}{2}\begin{pmatrix} (e^{t}+e^{-t}) \\ (e^{t}-e^{-t}) \end{pmatrix}} \quad \dots (12)$$

$$\Phi(t-\tau)\cdot B = \begin{pmatrix} \phi_{11}(t-\tau) & \phi_{12}(t-\tau) \\ \phi_{21}(t-\tau) & \phi_{22}(t-\tau) \end{pmatrix}\cdot\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} =$$

$$= \begin{pmatrix} b_1\cdot\phi_{11}(t-\tau) + b_2\cdot\phi_{12}(t-\tau) \\ b_1\cdot\phi_{21}(t-\tau) + b_2\cdot\phi_{22}(t-\tau) \end{pmatrix}$$

$$\boxed{\int_0^t \phi_{11}(t-\tau)\, d\tau = \int_0^t \phi_{22}(t-\tau)\, d\tau} = \frac{1}{2}\cdot\left\{\int_0^t e^{t-\tau} d\tau + \int_0^t e^{-t+\tau} d\tau\right\} =$$

$$= \frac{1}{2}\cdot\left\{e^{t}\cdot\int_0^t e^{-\tau} d\tau + e^{-t}\cdot\int_0^t e^{\tau} d\tau\right\} =$$

$$= \frac{1}{2}\cdot\left\{e^{t}\cdot(-e^{-\tau})\Big|_0^t + e^{-t}\cdot e^{\tau}\Big|_0^t\right\} =$$

$$= \frac{1}{2}\cdot\left\{-e^{t}\cdot(e^{-t}-1) + e^{-t}\cdot(e^{t}-1)\right\} =$$

$$= \frac{1}{2}\cdot\left\{-\cancel{1} + e^{+t} + \cancel{1} - e^{-t}\right\} = \boxed{\frac{1}{2}\cdot(e^{t}-e^{-t})} \quad \dots (13)$$

$$\boxed{\int_0^t \phi_{21}(t-\tau)\, d\tau = \int_0^t \phi_{12}(t-\tau)\, d\tau} = \frac{1}{2}\left\{\int_0^t e^{t-\tau} d\tau - \int_0^t e^{-t+\tau} d\tau\right\} =$$

$$= \frac{1}{2}\cdot\left\{e^{t}\cdot\int_0^t e^{-\tau} d\tau - e^{-t}\cdot\int_0^t e^{\tau} d\tau\right\} =$$

$$= \frac{1}{2}\cdot\left\{e^{t}\cdot(-e^{-\tau})\Big|_0^t - e^{-t}\cdot e^{\tau}\Big|_0^t\right\} =$$

$$= \frac{1}{2}\cdot\left\{-e^{t}\cdot(e^{-t}-1) - e^{-t}\cdot(e^{t}-1)\right\} =$$

$$= \frac{1}{2}\cdot\left\{-1 + e^{t} - 1 + e^{-t}\right\} =$$

$$= \boxed{\frac{1}{2}\cdot\left\{-2 + e^{t} + e^{-t}\right\}} \quad \ldots (14) \Rightarrow$$

$$\boxed{\int_0^t \Phi(t-\tau)\cdot B\cdot d\tau} = \frac{1}{2}\cdot\begin{pmatrix} b_1\cdot(e^{t}-e^{-t}) + b_2\cdot(-2+e^{t}+e^{-t}) \\ b_1\cdot(-2+e^{t}+e^{-t}) + b_2(e^{t}-e^{-t}) \end{pmatrix} =$$

$$\boxed{= \begin{pmatrix} \frac{1}{2}\cdot\left\{-2b_2 + (b_1+b_2)e^{t} + (-b_1+b_2)e^{-t}\right\} \\ \frac{1}{2}\left\{-2b_1 + (b_1+b_2)e^{t} + (b_1-b_2)e^{-t}\right\} \end{pmatrix}} \ldots (15)$$

$[(12),(15)] \rightarrow (11) \Rightarrow$

$$x_1(t) = \frac{1}{2}\cdot\left\{-2b_2 + (1+b_1+b_2)e^{t} + (1-b_1+b_2)e^{-t}\right\} \ldots (16)$$

$$x_2(t) = \frac{1}{2}\left\{-2b_1 + (1+b_1+b_2)e^{t} - (1-b_1+b_2)e^{-t}\right\} \ldots (17)$$

$[(16),(17)] \rightarrow (10) \Rightarrow$

$$\frac{1}{2}\cdot\tau_1\cdot\left\{-2b_1 - 2b_2 + 2(1+b_1+b_2)e^{t}\right\} + \frac{1}{2}\left\{-2b_1 + (1+b_1+b_2)e^{t} - (1-b_1+b_2)e^{-t}\right\} = 0 \Big| \cdot$$

$$\Rightarrow \boxed{-2\cdot(b_1 + b_1\tau_1 + b_2\tau_1) + \left\{(1+b_1+b_2)(2\tau_1+1)\right\}e^{t} - (1-b_1+b_2)e^{-t} = 0}$$

This expresion is surely equal to zero in the case that:

$$b_1 + b_1\tau_1 + b_2\tau_1 = 0 \quad \ldots (18)$$

$$(1+b_1+b_2)(2\tau_1+1) = 0 \quad \ldots (19)$$

$$1 - b_1 + b_2 = 0 \quad \ldots (20)$$

$(20) \to \quad b_2 = b_1 - 1 \quad \ldots (21)$

$(21) \to (19) \to \quad (1 + b_1 + b_1 - 1)(2\tau_1 + 1) = 0 \Rightarrow$

$\Rightarrow 2 b_1 (2\tau_1 + 1) = 0 \Leftrightarrow \quad b_1 = 0 \quad \ldots (22)$

$\vee$

$\tau_1 = -\frac{1}{2} \quad \ldots (23)$

$(22) \to (21) \Rightarrow \quad b_2 = -1 \quad \ldots (24)$

$[(22),(24)] \to (18) \Rightarrow \quad 0 + 0 \cdot \tau_1 - \tau_1 = 0 \Rightarrow \boxed{\tau_1 = 0} \ldots (25)$

$(25) \to (9) \to \quad \tau_2 = 0 + 1 \Rightarrow \quad \boxed{\tau_2 = 1} \ldots (26)$

$(22),(24),(25),(26) \Rightarrow$

$$\Rightarrow \boxed{B = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \; ; \; C = \begin{pmatrix} 0 & 1 \end{pmatrix}}$$

$(23) \to (18) \to \quad b_1 - \frac{1}{2} b_1 - \frac{1}{2} b_2 = 0 \,|\cdot 2 \Rightarrow \quad b_2 = 2b_1 - b_1$

$\boxed{b_2 = b_1} \ldots (27)$

(21) and (27) are in collision $\Rightarrow$

$\Rightarrow \tau_1$ can not be equal to $-\frac{1}{2}$! $\Rightarrow$

$$\Rightarrow \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \; B = \begin{pmatrix} 0 \\ -1 \end{pmatrix} ; \; C = \begin{pmatrix} 0 & 1 \end{pmatrix}$$