Due Wednesday 10/01/08

1. In the previous homework sets you studied the mass-spring system. Among other things, you developed the mathematical model and found the equilibrium points.

   (a) Study stability properties of these equilibrium points. Interpret your results from a physical point of view.

2. (**Floquet Theory**). Consider the system

$$\dot{x}(t) = A(t)x(t)$$

   where $A(t)$ is a periodic matrix with period T, so $A(t+T) = A(t)$. We want to study the state transition matrix $\Phi(t, t_0)$ associated with this periodically time-varying system.

   (a) First let us start with the state transition matrix $\Phi(t, 0)$, which satisfies

$$\begin{aligned} \dot{\Phi} &= A(t)\Phi \\ \Phi(0,0) &= I. \end{aligned}$$

   Define the matrix $\Psi(t,0) = \Phi(t+T,0)$ and show that Ψ satisfies

$$\begin{aligned} \dot{\Psi}(t,0) &= A(t)\Psi(t,0) \\ \Psi(0,0) &= \Phi(T,0). \end{aligned}$$

   (b) Show that this implies that $\Phi(t+T,0) = \Phi(t,0)\Phi(T,0)$.

   (c) Using Jacobi-Liouville formula, show that $\Phi(T,0)$ is invertible and therefore can be written as $\Phi(T,0) = \mathrm{e}^{TR}$.

   (d) Define

$$P(t)^{-1} = \Phi(t,0)\mathrm{e}^{-tR},$$

   and show that $P(t)^{-1}$, and consequently $P(t)$, are periodic with period T. Also show that $P(T) = I$. This means that

$$\Phi(t,0) = P(t)^{-1}\mathrm{e}^{tR}.$$

   (e) Show that $\Phi(0,t_0) = \Phi^{-1}(t_0,0)$. Using the fact that $\Phi(t,t_0) = \Phi(t,0)\Phi(0,t_0)$, show that

$$\Phi(t,t_0) = P(t)^{-1}\mathrm{e}^{(t-t_0)R}P(t_0).$$

   What is the significance of this result?

3. Consider the nonlinear system

$$\begin{aligned} \dot{x}_1 &= -x_1^2 + x_1x_2 \\ \dot{x}_2 &= -2x_2^2 + x_2 - x_1x_2 + 2 \end{aligned}$$

   (a) Show that $\bar{x} = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ is an equilibrium point.

   (b) Is $\bar{x}$ the only equilibrium state?

   (c) Verify that $\bar{x}$ is an asymptotically stable solution of the above system of differential equations.

   (d) Is it possible that $\bar{x}$ is a *globally* asymptotically stable equilibrium point?

4. The unforced Duffing oscillator is described by the following dynamical system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 - x_1^3 - \delta x_2, \qquad \delta > 0. \end{aligned}$$

   (a) Find the equilibrium points of the system.

   (b) Investigate their stability properties.

   Bonus points: Plot the phase portrait of this system and contemplate the possible limit cycles. You may use the quiver command in MATLAB.

5. Investigate the stability of the origin of the system

$$\begin{aligned} \dot{x}_1 &= (x_1 - x_2)(x_1^2 + x_2^2 - 1) \\ \dot{x}_2 &= (x_1 + x_2)(x_1^2 + x_2^2 - 1), \end{aligned}$$

   using a quadratic Lyapunov function.

6. Let us suppose that we perform the following tests.

   Test 1: Let $u(t) = 0$ for all t. When the initial state $x(0)$ is given by

$$x(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad \text{then} \qquad x(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \mathrm{e}^{-t}$$

   and $y(t) = -\mathrm{e}^{-t}$
   When the initial state is given by

$$x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \text{then} \qquad x(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mathrm{e}^{t}$$

   Test 2: Now suppose that for

$$u(t) = 1 \quad \text{for all } t \geq 0$$

   and for the specific initial state

$$x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

   we find that the output is identically zero for all time i.e.,

$$y(t) = 0, \quad \text{for all } t \geq 0.$$

   In each of the questions to follow, give the numerical answer called for, together with a brief explanation.

   (a) Write an explicit expression (not the infinite series) for the matrix exponential e^{tA}.

   (b) Suppose that $u(t) = 0$ for all $t \geq 0$; and that

$$x(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

   What is $x(t)$ for all $t \geq 0$?

   (c) Find the numerical values of state-space matrices A, B, and C.