EE 5231 HW4 solution

1. a) $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$$A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{thus} \quad A^k = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \quad \forall k \geq 2$$

$$e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!} = I + \frac{At}{1!} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

b) $A' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad A^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

so A^k is periodic with a period of 4.

$$e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!} = \begin{pmatrix} a_1(t) & a_2(t) \\ a_3(t) & a_4(t) \end{pmatrix}$$

$$a_1(t) = 1 + \frac{0 \cdot t}{1!} - \frac{1 \cdot t^2}{2!} + \frac{0 \cdot t^3}{3!} + \frac{1 \cdot t^4}{4!} + \cdots = \cos t$$

$$a_2(t) = 0 + \frac{1 \cdot t}{1!} + \frac{0 \cdot t^2}{2!} - \frac{t^3}{3!} + \frac{0 \cdot t^4}{4!} + \cdots = \sin t$$

$$a_3(t) = -\sin t, \quad a_4(t) = \cos t$$

$$e^{At} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

2. (a) $H(s) = \frac{1}{s^2}$ has multiple roots on imaginary axis, i.e. $s_1 = s_2 = 0$.

The E.P. is not stable.

(b) $\dot{x}_1 = x_2$

$$\dot{x}_2 = -\frac{c}{m} \cdot \frac{\bar{u}^2}{x_1^2} + g$$

E.P. $x_2 = 0$, $-\frac{c}{m} \cdot \frac{\frac{mgY}{c}}{x_1^2} + g = 0 \Rightarrow x_1^2 = Y$

EP_1 : $\begin{pmatrix} \sqrt{Y} \\ 0 \end{pmatrix}$, EP_2 : $\begin{pmatrix} -\sqrt{Y} \\ 0 \end{pmatrix}$

Use indirect method :

1° Linearize at EPs

$$A_1 = \left. \begin{pmatrix} 0 & 1 \\ 2 \cdot \frac{c}{m} \cdot \frac{\bar{u}^2}{x_1^3} & 0 \end{pmatrix} \right|_{x_1 = \sqrt{Y}} = \begin{pmatrix} 0 & 1 \\ \frac{2g}{\sqrt{Y}} & 0 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 0 & 1 \\ \frac{-2g}{\sqrt{Y}} & 0 \end{pmatrix}$$

2° Check stability

$\det(sI - A_1) = \det \begin{pmatrix} s & -1 \\ \frac{-2g}{\sqrt{Y}} & s \end{pmatrix} = s^2 - \frac{2g}{\sqrt{Y}}$ is unstable $\Rightarrow$ original system unstable.

$\det(sI - A_2) = s^2 + \frac{2g}{\sqrt{Y}}$, this linearized system is stable i.s.L.

But nothing can be concluded about stability of the nonlinear system's E.P. using indirect Lyapunov method.

3. $\dot{x}_1 = x_2$

$\dot{x}_2 = -\nu\left(e^{-x_1} - e^{-2x_1}\right)$, $\nu \neq 0$

Solution : (a) $x_2 = 0$, $-\nu\left(e^{-x_1} - e^{-2x_1}\right) = 0$ $\Rightarrow$ $x_1 = 0$, $x_2 = 0$

(b) Linearize at origin

$$A = \left.\begin{pmatrix} 0 & 1 \\ \mu\left(e^{-x_1} - 2e^{-2x_1}\right) & 0 \end{pmatrix}\right|_{x_1=0} = \begin{pmatrix} 0 & 1 \\ -\nu & 0 \end{pmatrix}$$

$$\det(sI - A) = \det\begin{pmatrix} s & -1 \\ \nu & s \end{pmatrix} = s^2 + \nu$$

1°, if $\nu < 0$, then the system has E-value on RHP, unstable.

2°, if $\nu > 0$, two E-values on imaginary axis.

The linearized system is stable i.s.L. at origin, but we can't claim stability of the original non-linear system.

Try Lyap function of the form :

$$V(x) = x_2^2 + \nu\left(1 - e^{-x_1}\right)^2$$

$$\dot{V}(x) = 2x_2 \cdot \dot{x}_2 + 2\mu\left(1 - e^{-x_1}\right) \cdot e^{-x_1} \cdot \dot{x}_1$$

$$= 2x_2 \cdot \left(-\nu\left(e^{-x_1} - e^{-2x_1}\right)\right) + 2\mu x_2\left(e^{-x_1} - e^{-2x_1}\right)$$

$$= 2(\mu - \nu)\, x_2\left(e^{-x_1} - e^{-2x_1}\right)$$

$$= 0$$

Thus, this E.P. is stable i.s.L.

[redacted]

$$2^\circ \quad \left.\begin{aligned} \dot{x}_1 &= -\frac{x_2}{1+x_1^2} - 2x_1 \\ \dot{x}_2 &= \frac{x_1}{1+x_1^2} \end{aligned}\right\} \; (*) \iff \underline{\dot{x}} = \underline{f}(\underline{x})$$

a) E.P.

$$-\frac{x_2}{1+x_1^2} - 2x_1 = 0 \quad \ldots (1)$$

$$\frac{x_1}{1+x_1^2} = 0 \quad \ldots (2) \rightarrow x_1 = 0 \quad \ldots (3)$$

$(3) \rightarrow (1) \Rightarrow x_2 = 0 \Rightarrow \underline{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a candidate for E.P. of the system $(*)$. Since $\underline{f}(\underline{x}) \in C^{(1)}(\mathbb{R} \times \mathbb{R}) \Rightarrow$

$\Rightarrow \underline{x} = \underline{0}_x$ is equilibrium point of the system $(*)$.

b) $v(\underline{x}) = x_1^2 + x_2^2$

Properties of $v(\underline{x})$:

$$\left.\begin{aligned} &1^\circ \; v(\underline{x}) \in C(\mathbb{R}^2) \\ &2^\circ \; v(\underline{0}_x) = 0 \\ &3^\circ \; v(\underline{x}) > 0, \; \forall \underline{x} \in \mathbb{R}^2 \setminus \{\underline{0}\} \\ &4^\circ \; v(\underline{x}) \to +\infty, \text{ as } \|\underline{x}\| \to \infty \end{aligned}\right\} \Rightarrow$$

$v(\underline{x})$ is positive definite on $\mathbb{R}^2$ + $v(\underline{x})$ is radially unbounded.

$$\boxed{\dot{v}(\underline{x})} = \frac{\partial v}{\partial x_1} \cdot \dot{x}_1 + \frac{\partial v}{\partial x_2} \cdot \dot{x}_2 = 2x_1 \dot{x}_1 + 2x_2 \dot{x}_2 =$$

$$= 2x_1 \cdot \left\{ -\frac{x_2}{1+x_1^2} - 2x_1 \right\} + 2x_2 \cdot \frac{x_1}{1+x_1^2} =$$

$$= -\frac{2x_1 x_2}{1+x_1^2} - 4x_1^2 + \frac{2x_1 x_2}{1+x_1^2} = \boxed{-4x_1^2} \Rightarrow$$

$$\left.\begin{aligned} x_2 &\neq 0 \\ x_1 &= 0 \end{aligned}\right\} \Rightarrow \dot{v}(\underline{x}) = 0$$

$\Rightarrow \dot{v}(\underline{x}) \leq 0, \; \forall \underline{x} \in \mathbb{R}^2 \Rightarrow v(\underline{x})$ is Lyapunov funct of the system $(*)$. $\Rightarrow \underline{x} = \underline{0}_x$ is stable equilibrium point in the sense of Lyapunov.

c) $f_1(x_1, x_2) = -\dfrac{x_2}{1+x_1^2} - 2x_1$

$f_2(x_1, x_2) = \dfrac{x_1}{1+x_1^2}$

$$A = \begin{pmatrix} \left.\dfrac{\partial f_1}{\partial x_1}\right|_0 & \left.\dfrac{\partial f_1}{\partial x_2}\right|_0 \\ \left.\dfrac{\partial f_2}{\partial x_1}\right|_0 & \left.\dfrac{\partial f_2}{\partial x_2}\right|_0 \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\frac{\partial f_1}{\partial x_1} = -2 - x_2 \cdot \frac{0 - 2x_1}{(1+x_1^2)^2} = -2 + \frac{4 x_1 x_2}{(1+x_1^2)^2} \Rightarrow$$

$$\Rightarrow \boxed{\left.\frac{\partial f_1}{\partial x_1}\right|_0 =} -2 + \frac{4 \cdot 0 \cdot 0}{(1+0^2)^2} \boxed{= -2}$$

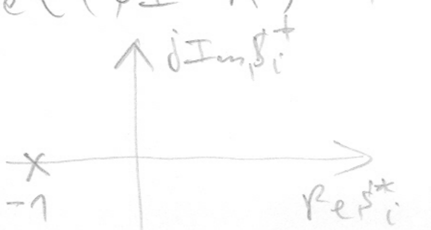
$$\frac{\partial f_1}{\partial x_2} = -\frac{1}{1+x_1^2} \Rightarrow \boxed{\left.\frac{\partial f_1}{\partial x_2}\right|_0 = -1}$$

$$\frac{\partial f_2}{\partial x_1} = \frac{1+x_1^2 - x_1 \cdot 2x_1}{(1+x_1^2)^2} = \frac{1-x_1^2}{(1+x_1^2)^2}$$

$$\boxed{\left.\frac{\partial f_2}{\partial x_1}\right|_0 = 1} \qquad \boxed{\frac{\partial f_2}{\partial x_2} = 0}$$

d) $$(sI - A) = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} s+2 & 1 \\ -1 & s \end{pmatrix}$$

$$\det(sI - A) = s^2 + 2s + 1 = (s+1)^2 \Rightarrow \boxed{s_1^* = s_2^* = -1}$$



Since all eigenvalues of the matrix A are in the left half of the complex plane $s \Rightarrow \underline{x} = \underline{0}_x$ of the system (*) is locally asimptotically stable.

e) $A^T P + P \cdot A = -Q$

$$Q = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \qquad P = P^T = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} ; \; A = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \cdot \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} -2p_{11} + p_{12} & -2p_{12} + p_{22} \\ -p_{11} & -p_{12} \end{bmatrix} + \begin{bmatrix} -2p_{11} + p_{12} & -p_{11} \\ -2p_{12} + p_{22} & -p_{12} \end{bmatrix} =$$

$$= \begin{bmatrix} -4p_{11} + 2p_{12} & -p_{11} - 2p_{12} + p_{22} \\ -p_{11} - 2p_{12} + p_{22} & -2p_{12} \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \quad 2(p_{12} - 2p_{11}) = -2 \quad \ldots (\mathrm{I})$$

$$-p_{11} - 2p_{12} + p_{22} = 0 \quad \ldots (\mathrm{II})$$

$$-2p_{12} = -2 \rightarrow \boxed{p_{12} = 1} \quad \ldots (\mathrm{III})$$

$$\mathrm{III} \rightarrow (\mathrm{I}) \rightarrow \quad 1 - 2p_{11} = -1 \Rightarrow 2p_{11} = 2 \Rightarrow \boxed{p_{11} = 1} \quad \ldots (\mathrm{IV})$$

$$((\mathrm{III}), (\mathrm{IV})) \rightarrow (\mathrm{II}) \Rightarrow \boxed{p_{22} = p_{11} + 2p_{12} = 1 + 2 = 3} \Rightarrow$$

$$\Rightarrow P = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \quad \left. \begin{array}{l} \Delta_1 = 1 > 0 \\ \Delta_2 = 1 \cdot 3 - 1 \cdot 1 = 2 > 0 \end{array} \right\} \Rightarrow$$

$$\Rightarrow P = P^T > 0 \Rightarrow V(\underline{x}) = \underline{x}^T P \underline{x} \geq$$

$$V(\underline{x}) = (x_1 \quad x_2) \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1^2 + 2x_1 x_2 + 3x_2^2$$

We have already shown that $\underline{x} = \underline{0}_x$ is asymptotically stable equilibrium point of the system ⊛ $\Rightarrow$

$\Rightarrow$ there is no need for finding $\dot{V}(\underline{x})$.

~~For $\forall x_1 x_2 \in \Omega$, Ω is a neighborhood of origin, we can choose b sufficiently large s.t. $V(x) > 0$~~

~~For instance, we want $V(x) > 0$ in unit ball, $\{x \mid x_1^2 + x_2^2 \le 1\}$~~

~~then choose $b > \max\limits_{-1 \le x_1 \le 1} \mu |e^{-2x_1} - 2e^{-x_1}|$~~

~~Therefore this E.P. is stable i.s.L.~~

---

5. Solution: (a) $\Phi(t, t_0) = \exp\left(\int_{t_0}^{t} A(\sigma)\,d\sigma\right)$

1° $\Phi(t, t) = \exp\left(\int_{t}^{t} A(\sigma)\,d\sigma\right) = I$

2° $$\frac{\partial \Phi(t, t_0)}{\partial t} = \exp\left(\int_{t_0}^{t} A(\sigma)\,d\sigma\right) \cdot A(t)$$

since $$A(t) \int_{t_0}^{t} A(\sigma)\,d\sigma = \int_{t_0}^{t} A(\sigma)\,d\sigma \cdot A(t) \quad , \cdots\cdots \quad (*)$$

$$A(t) \cdot \exp\left(\int_{t_0}^{t} A(\sigma)\,d\sigma\right) = \exp\left(\int_{t_0}^{t} A(\sigma)\,d\sigma\right) \cdot A(t)$$ by definition matrix exponenti

thus, $$\frac{\partial \Phi(t, t_0)}{\partial t} = A(t) \cdot \Phi(t, t_0)$$

(b) If $A(t)$ satisfies $(*)$, we can directly use the result from part (a).

Note $\alpha(t)$, $\beta(t)$ are continuous, then the integration on finite interval $[t_0, t]$ is well-defined.

Let $\int_{t_0}^{t} \alpha(\sigma)\,d\sigma = \bar{\alpha} - \bar{\alpha}_0 = \tilde{\alpha}$, $\int_{t_0}^{t} \beta(\sigma)\,d\sigma = \bar{\beta} - \bar{\beta}_0 = \tilde{\beta}$

$$\int_{t_0}^{t} A(\sigma)\,d\sigma = \begin{pmatrix} \tilde{\alpha} & \tilde{\beta} \\ -\tilde{\beta} & \tilde{\alpha} \end{pmatrix}$$

For notation simplicity, we omit $\tilde{\alpha}(t)$ as $\tilde{\alpha}$, but $\tilde{\alpha}$, $\tilde{\beta}$ are time-dependent.
similarly for α, β.

$$A(t)\int_{t_0}^{t} A(\sigma)\,d\sigma = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}\begin{pmatrix} \tilde{\alpha} & \tilde{\beta} \\ -\tilde{\beta} & \tilde{\alpha} \end{pmatrix} = \begin{pmatrix} \alpha\tilde{\alpha} - \beta\tilde{\beta} & \alpha\tilde{\beta} + \tilde{\alpha}\beta \\ -\tilde{\alpha}\beta - \alpha\tilde{\beta} & \alpha\tilde{\alpha} - \beta\tilde{\beta} \end{pmatrix}$$

$$\int_{t_0}^{t} A(\sigma)\,d\sigma \cdot A(t) = \begin{pmatrix} \tilde{\alpha} & \tilde{\beta} \\ -\tilde{\beta} & \tilde{\alpha} \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} = \begin{pmatrix} \alpha\tilde{\alpha} - \beta\tilde{\beta} & \alpha\tilde{\beta} + \tilde{\alpha}\beta \\ -\tilde{\alpha}\beta - \alpha\tilde{\beta} & \alpha\tilde{\alpha} - \beta\tilde{\beta} \end{pmatrix}$$

Thus, $\Phi(t, t_0) = \exp\left(\int_{t_0}^{t} A(\sigma)\,d\sigma\right)$