Due Wednesday 10/15/08

1. Use the (matrix) exponential series to evaluate e^{At} for:
   (a) $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$,
   (b) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

2. What kind of equilibrium stability (stable (in the sense of Lyapunov), or AS, or GAS) if any, is exhibited by the state representation of
   (a) The $\frac{1}{s^2}$ plant with no input, i.e. $\dot{x}_1 = x_2$, $\dot{x}_2 = 0$.
   (b) The magnetically suspended ball: $\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{-c}{m}\frac{\bar{u}^2}{x_1^2} + g \end{aligned}$ with $\bar{u} = \sqrt{\frac{mg}{c}Y} = \text{const}$.

3. The Morse oscillator is a model that is frequently used in chemistry to study reaction dynamics. The equations for an unforced Morse oscillator are given by

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\mu(e^{-x_1} - e^{-2x_1}). \end{aligned}$$

   (a) Find the equilibrium points of the system.
   (b) Investigate their stability properties.

4. Consider the following nonlinear system

$$\begin{aligned} \dot{x}_1 &= -\frac{x_2}{1+x_1^2} - 2x_1, \\ \dot{x}_2 &= \frac{x_1}{1+x_1^2}. \end{aligned}$$

   (a) Show that the origin is an equilibrium point.
   (b) Using the candidate Lyapunov function

$$V(x) = x_1^2 + x_2^2,$$

   what are the stability properties of the equilibrium point?
   (c) Linearize the nonlinear system around the equilibrium point.
   (d) What can you deduce about the stability properties of the origin using Lyapunov indirect method?
   (e) Obtain a suitable Lyapunov function by solving the Lyapunov equation

$$A^T P + PA = -Q,$$

   where

$$Q = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

5. (a) Suppose $A(t)$ is an $n \times n$ time-varying matrix with continuous entries that satisfies

$$A(t)\left(\int_{t_0}^{t} A(\sigma)d\sigma\right) = \left(\int_{t_0}^{t} A(\sigma)d\sigma\right)A(t).$$

   Show that the state transition matrix $\Phi(t_1, t_0)$ can be computed as

$$\Phi(t, t_0) = \exp\left(\int_{t_0}^{t} A(\sigma)d\sigma\right).$$

   (b) Find the state transition matrix $\Phi(t_1, t_0)$ for the matrix

$$A(t) = \begin{bmatrix} \alpha(t) & \beta(t) \\ -\beta(t) & \alpha(t) \end{bmatrix},$$

   where $\alpha(t)$ and $\beta(t)$ are continuous functions of t.