Due Thursday 10/13/11

1. For the following convection-diffusion equation on $L_2[-1, 1]$:

$$\begin{aligned} \phi_t(x,t) &= \phi_{xx}(x,t) \; - \; \phi_x(x,t) \; + \; u(x,t) \\ \phi(x,0) &= f(x) \\ \phi(\pm 1, t) &= 0 \end{aligned} \tag{1}$$

   (a) Determine the adjoint $\mathcal{A}^\dagger$ of the operator

$$\begin{cases} [\mathcal{A}\, f]\,(x) \;=\; \left[\dfrac{\mathrm{d}^2 f}{\mathrm{d}x^2} \;-\; \dfrac{\mathrm{d}f}{\mathrm{d}x}\right](x) \\ \mathcal{D}(\mathcal{A}) \;=\; \left\{ f \,\in\, L_2\,[-1,\,1],\; \dfrac{\mathrm{d}^2 f}{\mathrm{d}x^2} \,\in\, L_2\,[-1,\,1],\; f(\pm 1) \,=\, 0 \right\} \end{cases}$$

   with respect to the standard L_2 inner product.

   (b) Determine the eigenvalues and the corresponding eigenvectors of the operators $\mathcal{A}$ and $\mathcal{A}^\dagger$. (Please make sure that the eigenvectors are bi-orthogonal.)

   (c) If $\{v_n\}_{n \in \mathbb{N}}$ are the eigenfunctions of the operator $\mathcal{A}^\dagger$, determine the equation that governs the evolution of $\langle v_n, \phi \rangle$, where ϕ denotes the solution to (1).

   (d) Using the Riesz-spectral operator representation discussed in Lecture 9, determine the kernel function representing the operator that maps the input u into the output ϕ.

   (e) Plot the time dependence of the L_2 norm of ϕ for $\phi(x, 0) = 0$ and $u(x,t) = \delta(x)\delta(t)$.

2. The operator:

$$[\mathcal{A}\, g]\,(x) \;=\; \frac{1}{w(x)} \left(\frac{\mathrm{d}}{\mathrm{d}\,x} \left(p(x)\, \frac{\mathrm{d}\, g(x)}{\mathrm{d}\,x} \right) \;+\; q(x)\, g(x) \right),$$

   arises in many relevant problems in mathematical physics and it is known as the Sturm-Liouville operator. Let us assume that w, q, p, and p' are continuous functions on the interval $[a, b]$, with $p > 0$ and $w > 0$. We will endow $L_2[a, b]$ with the following *weighted* inner product

$$\langle f, g \rangle_w \;=\; \langle f, wg \rangle_2 \;=\; \int_a^b f(x)\, w(x)\, g(x)\, \mathrm{d}x.$$

   Show that the Sturm-Liouville operator is a self-adjoint operator, with respect to $\langle \cdot, \cdot \rangle_w$, for any combination of *homogeneous* Dirichlet and Neumann boundary conditions. (If you are more ambitious show that homogeneous Robin boundary conditions could be included in the mix as well.)

3. In this problem we will look at the equations that describe evolution of the "streamwise-constant" fluctuations in a channel flow of incompressible Newtonian fluids. For the geometry shown in Figure 1, (x, y, z) denote the spatial coordinates, and (u, v, w) are the velocity fluctuations in the corresponding spatial directions. The streamwise-constant linearized Navier-Stokes and continuity equations,

$$\begin{aligned} u_t &= -\,U' v \;+\; (1/Re)\,\Delta u, \\ v_t &= -\,p_y \;+\; (1/Re)\,\Delta v, \\ w_t &= -\,p_z \;+\; (1/Re)\,\Delta w, \\ 0 &= v_y \;+\; w_z, \end{aligned} \tag{LNSE}$$

   govern the dynamics of the velocity and pressure (p) fluctuations around the mean velocity $(U(y), 0, 0)$ and pressure $\bar{P}$. (In Problem 4 you will determine this flow as the the steady-state solution of the Navier-Stokes equations.) The non-dimensional parameter Re, which determines the ratio of inertial to viscous forces, denotes the Reynolds number, the subscripts denote spatial/temporal derivatives, and $U'(y) = \mathrm{d}U(y)/\mathrm{d}y$. All small fluctuations are assumed to evolve in the (y, z)-plane and they thus depend on y, z, and t; e.g., $u = u(y, z, t)$. Operator Δ denotes the two-dimensional Laplacian, $\Delta = \partial_{yy} + \partial_{zz}$.

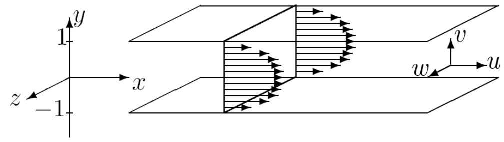

Figure 1: Channel flow geometry.

(a) System (LNSE) is not in the evolution form. This is because of the static-in-time constraint imposed by the continuity equation, $0 = v_y + w_z$. By introducing new variable ψ,

$$v \;=\; \psi_z, \quad w \;=\; -\,\psi_y,$$

show that

- continuity equation is automatically satisfied;
- the pressure can be eliminated from the system (LNSE), thereby bringing it to the evolution form with the state $\left[\begin{array}{cc} \psi & u \end{array}\right]^T$.

Hint: Differentiate the v-equation in (LNSE) with respect to z, and the w-equation with respect to y; subtract the latter from the former, and express (v, w) in terms of ψ.

(b) Use translational invariance in the z-direction (infinite spatial extent, constant coefficients) of the system obtained in Part (a) to bring it to the k_z-parameterized family of PDEs in y and t, where k_z is the wavenumber in the z-direction.

(c) If the velocity fluctuations satisfy homogeneous Dirichlet boundary conditions, i.e.,

$$\{\hat{u}(y = \pm 1, k_z, t) \;=\; 0,\; \hat{v}(y = \pm 1, k_z, t) \;=\; 0,\; \hat{w}(y = \pm 1, k_z, t) \;=\; 0\},$$

determine the boundary conditions on $\hat{\psi}$. Here, $\hat{u}$ denotes the field u obtained by applying the Fourier transform in the z-direction.

(d) Physically relevant quantity that provides measure of the "size" of velocity fluctuations is kinetic energy,

$$\begin{aligned} E(k_z, t) \;&=\; \frac{1}{2}\left(\langle \hat{u}, \hat{u}\rangle \;+\; \langle \hat{v}, \hat{v}\rangle \;+\; \langle \hat{w}, \hat{w}\rangle\right) \\ &=\; \frac{1}{2}\int_{-1}^{1} \left(\hat{u}^*(y, k_z, t)\,\hat{u}(y, k_z, t) \;+\; \hat{v}^*(y, k_z, t)\,\hat{v}(y, k_z, t) \;+\; \hat{w}^*(y, k_z, t)\,\hat{w}(y, k_z, t)\right) \mathrm{d}y. \end{aligned}$$

Express $E(k_z, t)$ in terms of $\hat{\psi}$ and $\hat{u}$. In other words, determine the inner product that gives energy in terms of the state $\left[\begin{array}{cc} \hat{\psi} & \hat{u} \end{array}\right]^T$.

Note: This will help us identify the "proper" function space for the evolution model that you determined in Part (a).

4. Determine the solutions to

$$\frac{1}{Re}\,U''(y) \;=\; -\,P_x,$$

for two different setups:

(a) $P_x = -2/Re$, $U(\pm 1) = 0$;

(b) $P_x = 0$, $U(\pm 1) = \pm 1$.

These determine the steady-state solutions of the Navier-Stokes equations in pressure- and shear-driven channel flows, respectively.

5. Determine the eigenvalue decomposition of the fourth derivative operator on $L_2[-1, 1]$ with homogeneous Dirichlet and Neumann boundary conditions at both ends (please use MATHEMATICA). Is this operator self-adjoint?