Due Friday 12/02/11

1. **Integral formulation of a differential equation**

Consider the following ordinary differential equation with boundary conditions

$$\left(D^{(2)} \,-\, \mathrm{i}\omega\right)\phi(y) \;=\; -\,d(y), \quad \omega \,\in\, \mathbb{R}, \tag{1a}$$

$$\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} E_{-1} \,+\, \begin{bmatrix} 0 \\ 1 \end{bmatrix} E_{1}\right)\phi(y) \;=\; \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{1b}$$

where $D^{(2)}$ is the second derivative operator, i is the imaginary unit, and E_{-1} and E_1 denote the point evaluation functionals at the boundaries, e.g.,

$$E_{-1}\,\phi\,(y) \;=\; \phi(-1).$$

System (1) is obtained by applying the temporal Fourier transform to externally forced diffusion equation on $L_2[-1,1]$ with Dirichlet boundary conditions, and it can be brought into an equivalent integral equation by introducing an auxiliary variable

$$\nu(y) \;=\; D^{(2)}\phi(y). \tag{2}$$

Integration of (2) yields

$$\begin{aligned} \phi'(y) \;&=\; \int_{-1}^{y} \nu(\eta_1)\,\mathrm{d}\eta_1 \;+\; k_1 \;=\; \left[J^{(1)}\,\nu\right](y) \;+\; k_1, \\ \phi(y) \;&=\; \int_{-1}^{y}\left(\int_{-1}^{\eta_2} \nu(\eta_1)\,\mathrm{d}\eta_1\right)\mathrm{d}\eta_2 \;+\; k_1\,(y\,+\,1) \;+\; k_2 \\ &=\; \left[J^{(2)}\,\nu\right](y) \;+\; K^{(2)}\,\mathbf{k}, \end{aligned} \tag{3}$$

where $J^{(1)}$ and $J^{(2)}$ denote the indefinite integration operators of degrees one and two, the vector $\mathbf{k} = \begin{bmatrix} k_2 & k_1 \end{bmatrix}^T$ contains the constants of integration which are to be determined from the boundary conditions (1b), and

$$K^{(2)} \;=\; \begin{bmatrix} 1 & (y+1) \end{bmatrix}.$$

The integral form of the 1D diffusion equation is obtained by substituting (3) into (1),

$$\left(I - \mathrm{i}\omega J^{(2)}\right)\nu(y) \;-\; \mathrm{i}\omega\,K^{(2)}\,\mathbf{k} \;=\; -\,d(y), \tag{4a}$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} k_2 \\ k_1 \end{bmatrix} \;+\; \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} E_{-1} \,+\, \begin{bmatrix} 0 \\ 1 \end{bmatrix} E_{1}\right) J^{(2)}\nu(y) \;=\; \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{4b}$$

Now, by observing that

$$E_{-1}J^{(1)}\nu(y) \;=\; \int_{-1}^{-1} \nu(\eta)\,\mathrm{d}\eta \;=\; 0,$$

we can use (4b) to express the constants of integration $\mathbf{k}$ in terms of ν,

$$\begin{bmatrix} k_2 \\ k_1 \end{bmatrix} \;=\; -\frac{1}{2}\begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} E_1\,J^{(2)}\,\nu(y) \;=\; \begin{bmatrix} 0 \\ -1/2 \end{bmatrix} E_1\,J^{(2)}\,\nu(y). \tag{5}$$

Finally, substitution of (5) into (4a) yields an equation for ν,

$$\left(I \,-\, \mathrm{i}\omega J^{(2)} \,+\, \frac{1}{2}\mathrm{i}\omega\,(y\,+\,1)\,E_1 J^{(2)}\right)\nu(y) \;=\; -d(y). \tag{6}$$

System (6) only contains indefinite integration operators and point evaluation functionals which are known to be well-conditioned.

- Make sure that you understand the above derivation.

- Determine the solution to (1) with

$$d(y) \;=\; y \sin\left(\frac{1}{2}\pi\,(y+1)\right) \;+\; y^2, \quad \omega \;=\; 1,$$

  using:

  (a) MATLAB Differentiation Matrix Suite [Weidemann & Reddy (2000)];
  (b) Chebfun's differential operators; and
  (c) Chebfun's indefinite integration operators. See the attached m-file to see how to do this.

- Repeat items (a)-(c) above for the heat equation with Neumann boundary conditions.

2. Consider the following boundary value problem

$$\phi''''(y) - 2\cos(2y)\,\phi'''(y) + \left[48\cos^2(2y)\,(1+\sin(2y)) \;-\; 16\sin(2y)\,(1\;+\;3\sin(2y))\right]\phi(y) \;=\; 0,$$

with boundary conditions

$$\phi(0) = 1, \;\; \phi'(0) = 2, \;\; \phi'(2\pi) = 2, \;\; \phi''(2\pi) = 4.$$

- Determine the solution using

  (a) Chebfun's differential operators;
  (b) Chebfun's indefinite integration operators.

- Compare your solutions to (a) and (b) with the exact solution

$$\phi(y) = \exp\left(\sin\left(2y\right)\right).$$

- Compute the condition number of the underlying operators in (a) and (b) for different numbers of collocation points and comment on them.

3. **Singular values of the frequency response operator**

Let a one-dimensional diffusion equation with homogenous Dirichlet boundary conditions and zero initial conditions be subject to spatially and temporally distributed forcing $d(y,t)$,

$$\begin{aligned} \phi_t(y,t) &\;=\; \phi_{yy}(y,t) \;+\; d(y,t), \\ \phi(\pm 1, t) &\;=\; 0, \\ \phi(y,0) &\;=\; 0, \;\; y \in [-1,\,1]. \end{aligned} \tag{7}$$

Considering ϕ as the field of interest, the frequency response operator for this system (from input d to output ϕ) is given by

$$\mathcal{T}(\omega) \;=\; \left(\mathrm{i}\omega I \;-\; D^{(2)}\right)^{-1}, \tag{8}$$

where $D^{(2)}$ is the second derivative operator with homogenous Dirichlet boundary conditions, I is the identity operator, and ω is the temporal frequency. Alternatively, by applying the temporal Fourier transform on (7) we obtain the following input-output differential equation representing the frequency response operator

$$\mathcal{T}(\omega) : \left\{ \begin{array}{l} \phi''(y,\omega) \;-\; \mathrm{i}\omega\phi(y,\omega) \;=\; -d(y,\omega), \\ \phi(\pm 1, \omega) \;=\; 0. \end{array} \right.$$

(a) Convince yourself that the adjoint of (8) is given by

$$\mathcal{T}^\dagger(\omega) \;=\; -\left(\mathrm{i}\omega I \;+\; D^{(2)}\right)^{-1}. \tag{9}$$

Find the equivalent input-output differential equation for (9) and for the composition operator $\mathcal{T}\mathcal{T}^\dagger$.

(b) Determine the largest singular value of the frequency response operator $\mathcal{T}$,

$$\sigma^2_{\max}(\mathcal{T}) \;=\; \lambda_{\max}(\mathcal{T}\mathcal{T}^\dagger),$$

as a function of the temporal frequency ω using MATLAB Differentiation Matrix Suite.

(c) Find the equivalent integral formulation of the input-output differential equations for the operator $\mathcal{T}\mathcal{T}^\dagger$.

(d) Write a program to compute the largest singular value of $\mathcal{T}$ as a function of ω using Chebfun indefinite integration operators.

Hint: *You will need the following command to construct operators with inverses*:

```
% Realizing an operator T = A^{-1}*B, where A and B are linear operators
% in Chebfun
        dom = domain(-1,1);  % domain of functions
% n-by-n realization
mat = @(n) A(n) \ B(n);
% functional expression
op = @(v) (A \ B)*vv;
% Realization of the operator T = A^{-1}*B
T = linop(mat,op,dom);
```

(e) Change the m-file Smax_HeatEq_inteigs to compute the largest singular value of $\mathcal{T}$ as a function of ω. Compare the results with the largest singular values computed in (b) and (d).

Note: *Make sure to have functions* Smax_FreqResp_inteigs.m *and* inteigs_system_TTs.m *in the directory in which you are doing your* MATLAB *computations.*

4. Consider the following boundary value problem

$$\begin{aligned} \phi_t(y,t) &= \phi_{yy}(y,t) \,-\, \big(\mathrm{i}k_x(1-y^2) \,+\, k_x^2\big)\,\phi(y,t) \,+\, d(y,t),\\ \phi(\pm 1,t) &= 0,\\ \phi(y,0) &= 0, \;\; y \in [-1,\,1], \end{aligned}$$

where k_x is the real number.

- Find the input-output differential equation representing the frequency response operator.
- Compute the largest singular values of the frequency response operator as a function of ω for several nonzero values of k_x using function Smax_FreqResp_inteigs.m. Comment on your results and compare them with $k_x = 0$ computations (standard diffusion equation).
- Compute the H_2 norm of your system (from d to ϕ) as a function of k_x.
- Determine the value of k_x at which the H_2 peaks and compute the principal eigenfunction of the controllability Gramian at this value of k_x.