Lectures 2 & 3: Examples of distributed systems

- Simple PDEs
  - Diffusion equation
  - Linear transport equation
  - Wave equation
  - Evolution of population equation

- Not-so-simple PDEs
  - Reaction-diffusion equation
  - Swift-Hohenberg equation
  - Navier-Stokes equations
- Networks of dynamic systems
  - Coordinated/cooperative control
  - Leader selection in dynamic networks
  - Micro-cantilever arrays
  - Biochemical networks
  - Wind farms

- Distributed control
  - Feedback-based
  - Sensor-free
Diffusion equation

\[
\begin{align*}
\frac{\partial \phi(x, t)}{\partial t} &= \frac{\partial^2 \phi(x, t)}{\partial x^2} + u(x, t) \iff \phi_t(x, t) &= \phi_{xx}(x, t) + u(x, t)
\end{align*}
\]

\(\phi(x, t)\) – temperature at position \(x\) and time \(t\)

\(u(x, t)\) – heat addition along the bar

- Need to specify initial and boundary conditions

**One IC:**

\[
\phi(x, 0) = \phi_0(x)
\]

**Homogeneous Dirichlet:**

\[
\phi(\pm 1, t) = 0
\]

**Homogeneous Neumann:**

\[
\phi_x(\pm 1, t) = 0
\]

**Homogeneous Robin:**

\[
\begin{align*}
ax(-1, t) + b \phi_x(-1, t) &= 0 \\
c \phi(+1, t) + d \phi_x(+1, t) &= 0
\end{align*}
\]
• In higher spatial dimensions

\[ \phi_t(x, t) = \Delta \phi(x, t) + u(x, t) \]

\[ x = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^T \] – vector of spatial coordinates

\[ \Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \] – Laplacian

• Boundary actuation in 1D

\[ \phi_t(x, t) = \phi_{xx}(x, t) + d(x, t) \]

\[ \phi(x, 0) = \phi_0(x) \]

\[ \phi(-1, t) = u(t), \ \phi(+1, t) = 0 \]
A finite dimensional example

- Mass-spring system

\[ m\ddot{\phi}(t) + k\phi(t) = u(t) \]

\( \phi(t) \) – position of a mass at time \( t \)

\( u(t) \) – force acting on a mass

- A state-space representation

\[
\begin{bmatrix}
\dot{\psi}_1(t) \\
\dot{\psi}_2(t)
\end{bmatrix}
= \begin{bmatrix}
0 & 1 \\
-k/m & 0
\end{bmatrix}
\begin{bmatrix}
\psi_1(t) \\
\psi_2(t)
\end{bmatrix}
+ \begin{bmatrix}
0 \\
1/m
\end{bmatrix} u(t)
\]

\[ \phi(t) = \begin{bmatrix}
1 & 0
\end{bmatrix}
\begin{bmatrix}
\psi_1(t) \\
\psi_2(t)
\end{bmatrix} \]

\( \psi_1(t) = \phi(t) \) – position at time \( t \)

\( \psi_2(t) = \dot{\phi}(t) \) – velocity at time \( t \)
State-space (evolution) representation

\[ \dot{\psi}(t) = A \psi(t) + B u(t) \]
\[ \phi(t) = C \psi(t) \]

- Finite dimensional state space: \( \psi(t) \in \mathbb{R}^n \)

- Variations of constants formula

\[ \psi(t) = e^{At} \psi(0) + \int_0^t e^{A(t-\tau)} B u(\tau) \, d\tau \]

- Can we do something similar for infinite dimensional systems?
**Linear transport equation**

\[
\phi_t(x, t) = -a \phi_x(x, t)
\]

\[
\phi(x, 0) = f(x), \ x \in \mathbb{R}
\]

Spatial Fourier transform

\[
\hat{\phi}(\kappa, t) = \int_{-\infty}^{\infty} \phi(x, t) e^{-j\kappa x} \, dx
\]

yields

\[
\begin{align*}
\dot{\hat{\phi}}(\kappa, t) &= -(a \, j\kappa) \hat{\phi}(\kappa, t) \\
\hat{\phi}(\kappa, 0) &= \hat{f}(\kappa), \ \kappa \in \mathbb{R}
\end{align*}
\]

\[
\Rightarrow \hat{\phi}(\kappa, t) = e^{-a \, j\kappa \, t} \hat{f}(\kappa)
\]

- Back to physical space

\[
\phi(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\phi}(\kappa, t) e^{j\kappa x} \, d\kappa = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\kappa) e^{j\kappa(x - at)} \, d\kappa = f(x - at)
\]

Solution doesn’t appear to be of the form: 

\["e^{-a \, \partial_x}" \times f(x)\]
Diffusion equation

\[ \phi_t(x, t) = \phi_{xx}(x, t) + u(x, t) \]
\[ \phi(x, 0) = \phi_0(x) \]
\[ \phi(\pm 1, t) = 0 \]

Define \( \psi(t) = \phi(\cdot, t) \) and write an abstract evolution equation:

\[ \dot{\psi}(t) = \mathcal{A}\psi(t) + u(t) \]
\[ \phi(t) = \psi(t) \]

- Infinite dimensional state-space: \( \psi(t) \in H \)
A candidate for state-space square-integrable functions: \( \mathcal{H} = L_2[-1, 1] = \left\{ f, \int_{-1}^{1} f^*(x) f(x) \, dx < \infty \right\} \)

\( A = \frac{d^2}{dx^2} \) + boundary conditions (contained in the domain of \( A \))

\( \mathcal{D}(A) = \left\{ f \in L_2[-1, 1], \frac{d^2 f}{dx^2} \in L_2[-1, 1], f(\pm 1) = 0 \right\} \)
Wave equation

\[ \phi_{tt}(x, t) = \phi_{xx}(x, t) + u(x, t) \]
\[ \phi(x, 0) = \phi_{10}(x), \quad \phi_t(x, 0) = \phi_{20}(x), \]
\[ \phi(\pm 1, t) = 0 \]

Define \( \psi(t) = \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix} = \begin{bmatrix} \phi(\cdot, t) \\ \phi_t(\cdot, t) \end{bmatrix} \) and write an abstract evolution equation:

\[
\begin{bmatrix}
\dot{\psi}_1(t) \\
\dot{\psi}_2(t)
\end{bmatrix} = \begin{bmatrix}
0 & I \\
d^2/dx^2 & 0
\end{bmatrix} \begin{bmatrix}
\psi_1(t) \\
\psi_2(t)
\end{bmatrix} + \begin{bmatrix}
0 \\
I
\end{bmatrix} u(t)
\]

\[
\phi(t) = \begin{bmatrix}
I & 0
\end{bmatrix} \begin{bmatrix}
\psi_1(t) \\
\psi_2(t)
\end{bmatrix}
\]

\[
E(t) = \frac{1}{2} \int_{-1}^{1} (\phi^2_x(x, t) + \phi^2_t(x, t)) \, dx
\]
\[
= \frac{1}{2} \int_{-1}^{1} (\psi^2_{1x}(x, t) + \psi^2_2(x, t)) \, dx
\]

- Energy of a wave:

- Selection of state-space: more subtle than for diffusion equation!
Evolution of population equation

\[ \phi_t(x, t) = -\phi_x(x, t) - \mu(x, t) \phi(x, t) \]
\[ \phi(x, 0) = \phi_0(x) \quad x \geq 0, \]
\[ \phi(0, t) = u(t), \quad t \geq 0 \]

\( \phi(x, t) \) – number of people of age \( x \) at time \( t \)

\( \mu(x, t) \) – mortality function

\( \phi_0(x) \) – initial age distribution

\( u(t) \) – number of people born at time \( t \)

- Control problem: design \( u \) to achieve desired age profile \( \phi_d(x) \) at time \( T \)
**Reaction-diffusion equations**

\[ \phi_t(x, t) = D \Delta \phi(x, t) + f(\phi(x, t)) \]

\( \phi \) – vector-valued field of interest

\( f(\phi) \) – nonlinear reaction term

\( \Delta \) – Laplacian

\( D \) – matrix of positive diffusion constants

**MAPK cascades**: responsible for cell proliferation and growth

\[ \begin{align*}
\phi_{1t} &= 0.001 \phi_{1xx} - \frac{\phi_1}{1 + \phi_1} + \frac{0.4}{1 + \phi_3} \\
\phi_{2t} &= 0.001 \phi_{2xx} - \frac{\phi_2}{1 + \phi_2} + 0.4\phi_1 \\
\phi_{3t} &= 0.001 \phi_{3xx} - \frac{\phi_3}{1 + \phi_3} + 0.4\phi_2
\end{align*} \]
Swift-Hohenberg equation

$$\phi_t = \epsilon \phi - (\Delta + 1)^2 \phi + c \phi^2 - \phi^3$$

Nonlinear: first order in time, fourth order in space

- Web-site of Michael Cross at Caltech contains interactive demonstrations
Navier-Stokes equations

conservation of momentum: \( \mathbf{v}_t = - (\mathbf{v} \cdot \nabla) \mathbf{v} - \nabla p + \frac{1}{Re} \Delta \mathbf{v} + \mathbf{d} \)

conservation of mass: \( 0 = \nabla \cdot \mathbf{v} \)

Describe the fluid motion

Nonlinear system of equations for:

\[
\begin{cases}
\text{pressure:} & p(x_1, x_2, x_3, t) \\
\text{velocity:} & \mathbf{v} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^T
\end{cases}
\]

“del” operator: \( \nabla = \frac{\partial}{\partial x_1} \mathbf{e}_1 + \frac{\partial}{\partial x_2} \mathbf{e}_2 + \frac{\partial}{\partial x_3} \mathbf{e}_3 \)

Reynolds number: \( Re = \frac{\text{inertial forces}}{\text{viscous forces}} \)
Networks of dynamic systems

Coordinated control

Micro-cantilever arrays

Biochemical networks

Wind farms

Gene Regulation: Jacob & Monod ('61), Goodwin ('65), Elowitz & Leibler (2000)

Cellular Signaling: Kholodenko (2000), Shvartsman et al.

Kinase

M3

P

M3

M2

P

M2

M1

P

M1

Phosphatase

Phosphatase

Phosphatase

Phosphatase

Metabolic Pathways: Morales and McKay ('67), Stephanopoulos et al. ('98)
Feedback flow control

- Control-oriented modeling of turbulent flows
- Design of estimators for turbulent flows
- Design of spatially localized distributed controllers
- Design of controllers of low dynamical order
Flow control in nature . . .

. . . and in swimming competitions
Riblets

PDEs with spatially periodic coefficients
Blowing and suction along the walls

NORMAL VELOCITY: \( V(y = \pm 1) = \mp \alpha \cos (\omega_x (x - ct)) \)

- **TRAVELING WAVE PARAMETERS:**
  - spatial frequency: \( \omega_x \)
  - speed: \( c \) \( \begin{cases} c > 0 & \text{downstream} \\ c < 0 & \text{upstream} \end{cases} \)
  - amplitude: \( \alpha \)

- **INVESTIGATE THE EFFECTS OF** \( c, \omega_x, \alpha \) **ON:**
  - ★ cost of control
  - ★ onset of turbulence