Numerical Methods

11 - 01 - 11

Motivating example : $\mathcal{L} = \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}$ $\oplus$ $f(\pm 1) = 0$

Q: Kernel representation of $\mathcal{L}^{-1}$?

$$\begin{cases} Af = g \\ f(\pm 1) = 0 \end{cases} \Rightarrow \begin{cases} f = \mathcal{L}^{-1} g \\ f(\pm 1) = 0 \end{cases}$$

①

$$f_1 = f, \quad f_2 = f' \quad \Rightarrow \begin{bmatrix} f_1' \\ f_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} g$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} f_1(-1) \\ f_2(-1) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f_1(1) \\ f_2(1) \end{bmatrix}$$

$$f(x) = \int_{-1}^{1} K(x, \xi) g(\xi) \, d\xi$$

$$= \left[A^{-1} g \right](x)$$

② Do eigenvalue decomposition of $\mathcal{L}$ and use the fact that $\mathcal{L}$ is self-adjoint w.r.t.

$$\langle f, g \rangle_w = \int_{-1}^{1} f(x) \, g(x) \, e^{-x} \, dx$$

$$\mathcal{L} v_n = \lambda_n v_n$$

$$f(x) = \left[\mathcal{L}^{-1} g \right](x) =$$

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} v_n(x) \langle v_n, g \rangle_w =$$

$$\int_{-1}^{1} \underbrace{\sum_{n=1}^{\infty} v_n(x) \, v_n(\xi) \, e^{-\xi}}_{K(x,\xi)} g(\xi) \, d\xi$$

What should we do for problems that are not as simple, meaning that the operator $\mathcal{A}$ is such that eigenvalue decomposition of $\mathcal{A}$ is difficult.

Here, we use <u>tools</u> for <u>numerically</u> solving these problems.

$\boxed{\text{Spectral methods}}$ use <u>global</u> information to approximate derivative operators. We end up having <u>full</u> matrices. <u>Error</u> decays <u>exponentially</u> with the number of discritization points, $\mathcal{O}(c^{-N})$

<u>vs.</u>

$\boxed{\text{Finite-difference methods}}$ use local information. The underlying matrices are <u>sparse</u>. <u>Error</u> decays as $\mathcal{O}(N^{-p})$ where N is the number of discritization points, and p is an integer $p > 0$.

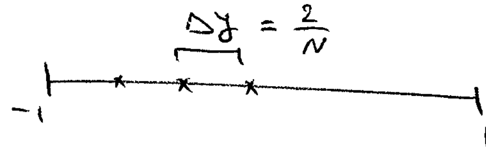
<u>vs.</u>

$\boxed{\text{Psuedo-spectral methods}}$ Accuracy similar to spectral methods. Computationally easier than spectral methods (Close to finite-differences).

Example

Operator $\quad U(y) \dfrac{d^2}{dy^2} \Psi$

- Finite-differences @ $\bar{y}$



$$U(\bar{y}) \frac{\Psi(\bar{y}+\Delta y) - 2\Psi(\bar{y}) + \Psi(\bar{y}-\Delta y)}{(\Delta y)^2} \rightarrow \mathcal{O}((\Delta y)^2) = \mathcal{O}\left(\frac{1}{N^2}\right)$$

In matrix form:

$$\mathrm{diag}(U) \cdot T \cdot \underline{\Psi}$$

$$T \downarrow \quad \text{Toeplitz} \quad \frac{1}{(\Delta y)^2}\begin{bmatrix} \ddots & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & 1 & -2 & \ddots \\ & & & \ddots & \ddots \end{bmatrix}$$

- In Psuedo-spectral:

$$\mathrm{diag}(U) \cdot D_2 \cdot \underline{\Psi}$$

$$\underline{\Psi} = \begin{bmatrix} \vdots \\ \Psi(\bar{y}-\Delta y) \\ \Psi(\bar{y}) \\ \Psi(\bar{y}+\Delta y) \\ \vdots \end{bmatrix}$$