Lecture 21: Input-output analysis in fluid mechanics

- Linear analyses: Input-output vs. Stability

AMPLIFICATION:

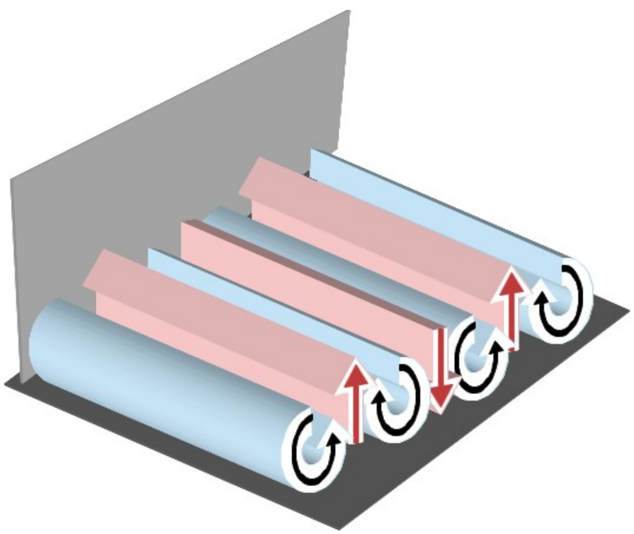
$$\mathbf{v} \;=\; \mathcal{T}\,\mathbf{d}$$

singular values of $\mathcal{T}$

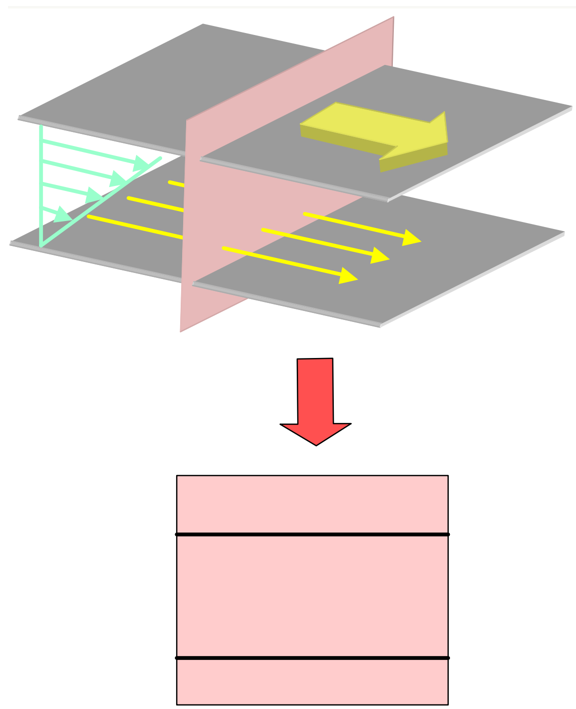
STABILITY:

$$\psi_t \;=\; \mathcal{A}\,\psi$$

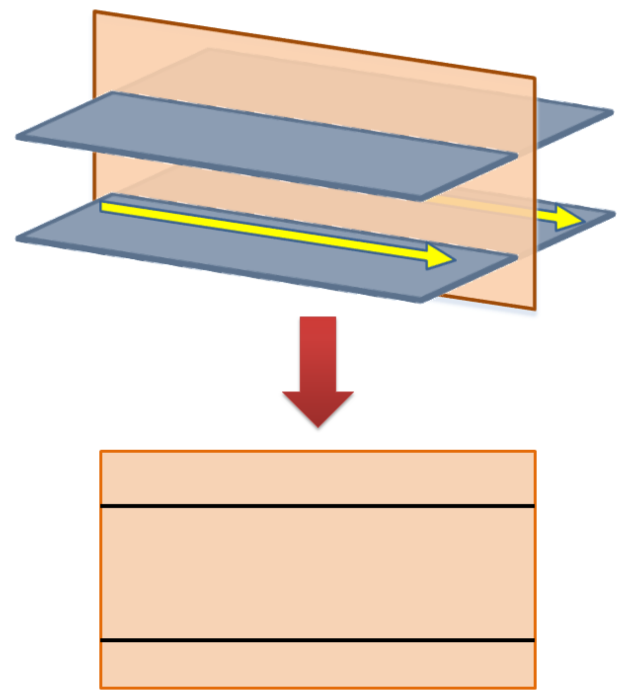
e-values of $\mathcal{A}$



typical structures



cross-sectional dynamics



2D models

Transition in Newtonian fluids

- LINEAR HYDRODYNAMIC STABILITY: **unstable normal modes**
  - ⋆ **successful in:** Benard Convection, Taylor-Couette flow, etc.
  - ⋆ **fails in:** wall-bounded shear flows (channels, pipes, boundary layers)

- DIFFICULTY #1

  Inability to predict: Reynolds number for the onset of turbulence (Re_c)

  Experimental onset of turbulence: { **much before instability** / **no sharp value for** Re_c }

- DIFFICULTY #2

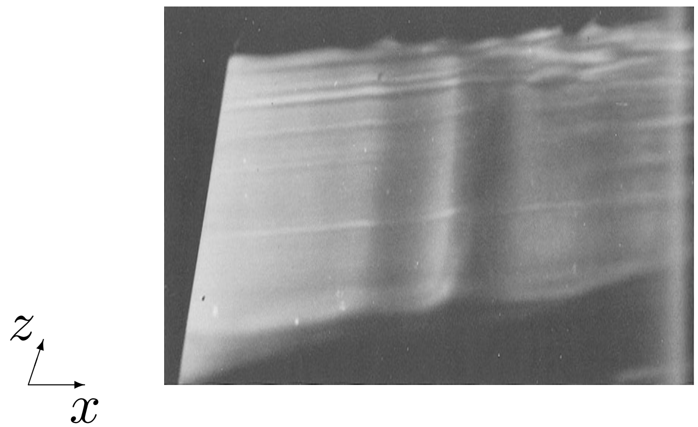
  Inability to predict: flow structures observed at transition

  (except in carefully controlled experiments)

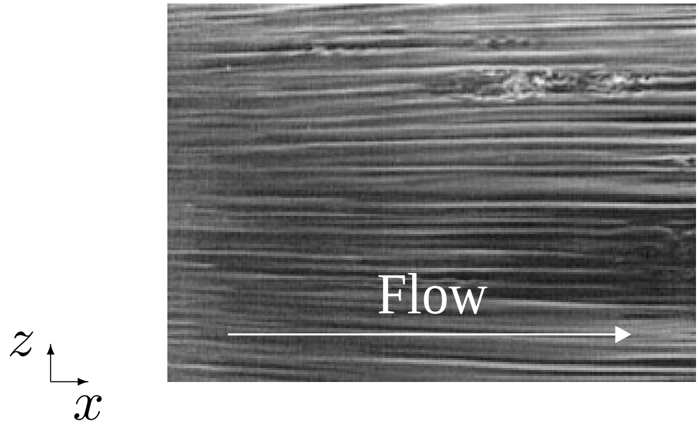
LINEAR STABILITY:

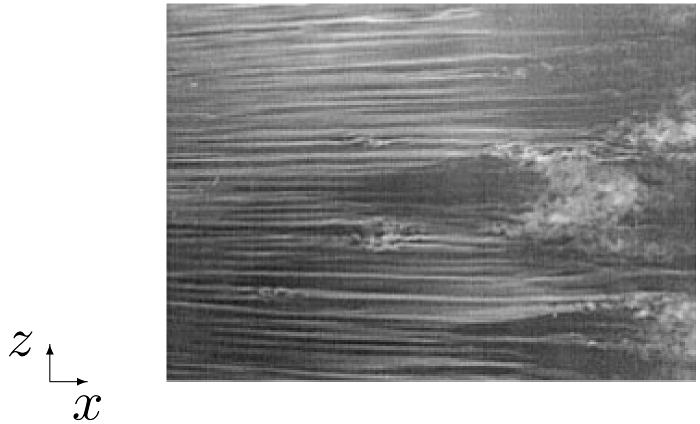
⋆ For $Re \geq Re_c \;\; \Rightarrow \;\;$ exp. growing normal modes

corresponding e-functions (TS-waves) } $:=$ exp. growing flow structures



EXPERIMENTS: **streaky boundary layers and turbulent spots**





Matsubara & Alfredsson, *J. Fluid Mech. '01*

- FAILURE OF LINEAR HYDRODYNAMIC STABILITY
  caused by high flow sensitivity
  - ⋆ large transient responses
  - ⋆ large noise amplification
  - ⋆ small stability margins

TO COUNTER THIS SENSITIVITY: **must account for modeling imperfections**

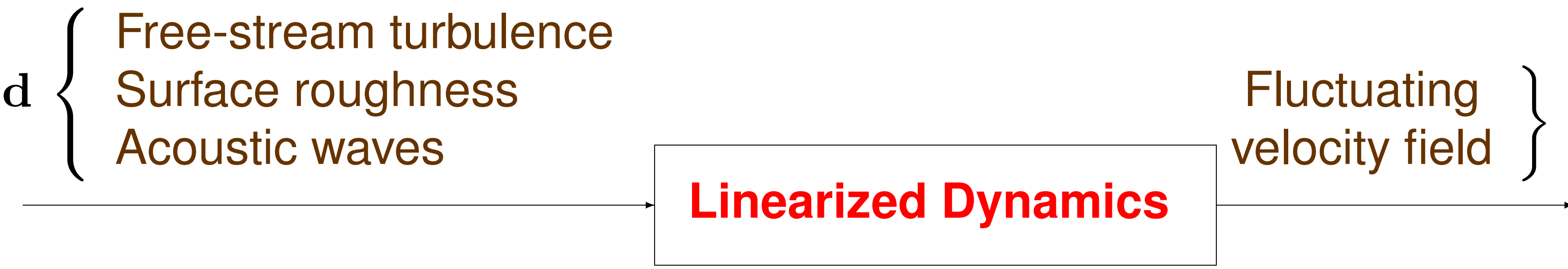
TRANSITION ≈ STABILITY + RECEPTIVITY + ROBUSTNESS

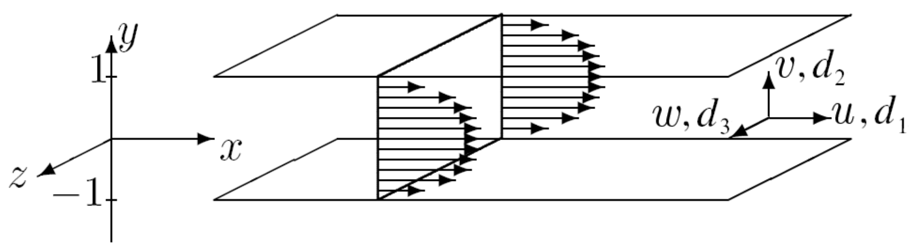
RECEPTIVITY → **flow disturbances**

ROBUSTNESS → **unmodeled dynamics**

Tools for quantifying sensitivity

- INPUT-OUTPUT ANALYSIS: **spatio-temporal frequency responses**





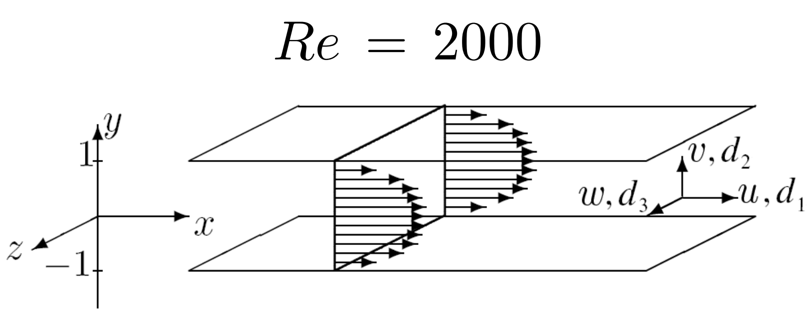
$$\underbrace{\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}}_{\mathbf{d}} \xrightarrow{\text{amplification}} \underbrace{\begin{bmatrix} u \\ v \\ w \end{bmatrix}}_{\mathbf{v}}$$

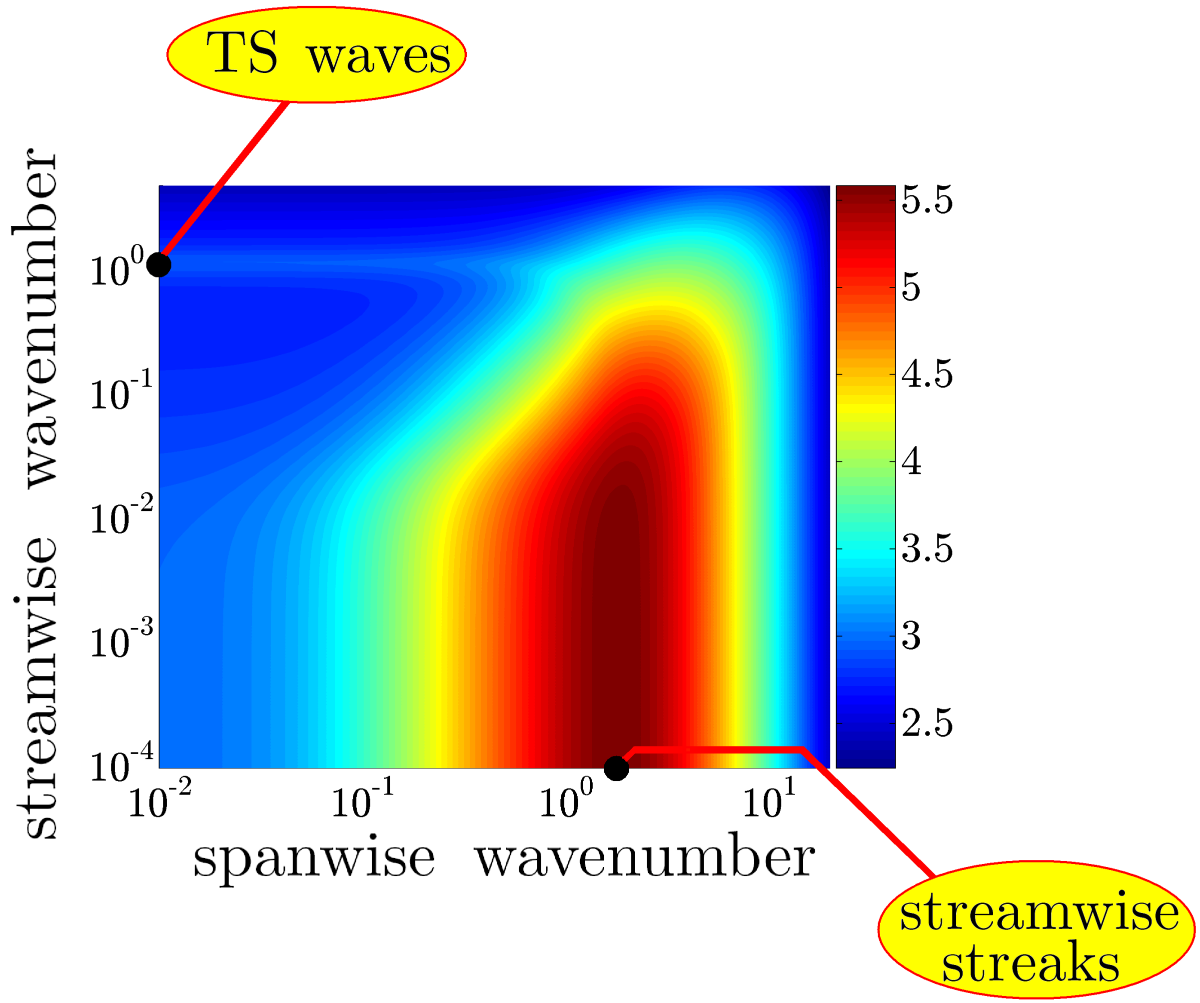
IMPLICATIONS FOR:

transition: insight into mechanisms

control: control-oriented modeling

Ensemble average energy density





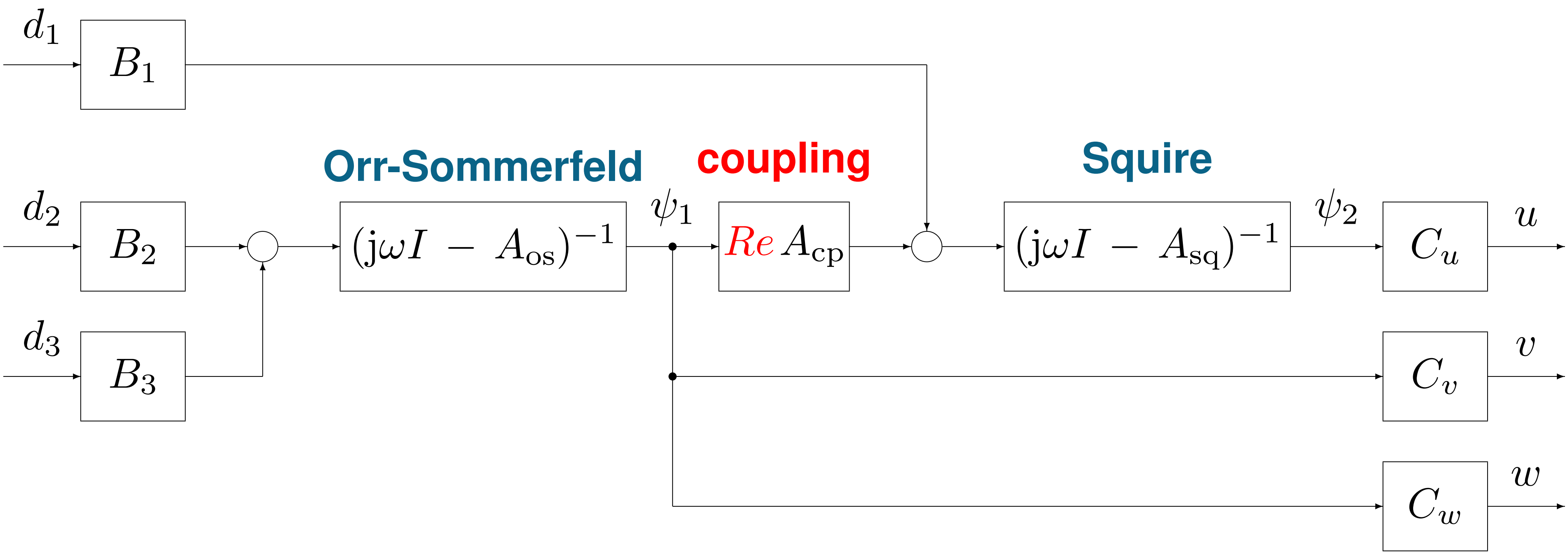
- **Dominance of streamwise elongated structures**

  streamwise streaks!

Influence of Re: streamwise-constant model

$$\begin{bmatrix} \psi_{1t} \\ \psi_{2t} \end{bmatrix} = \begin{bmatrix} A_{\rm os} & 0 \\ Re\, A_{\rm cp} & A_{\rm sq} \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} + \begin{bmatrix} 0 & B_2 & B_3 \\ B_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

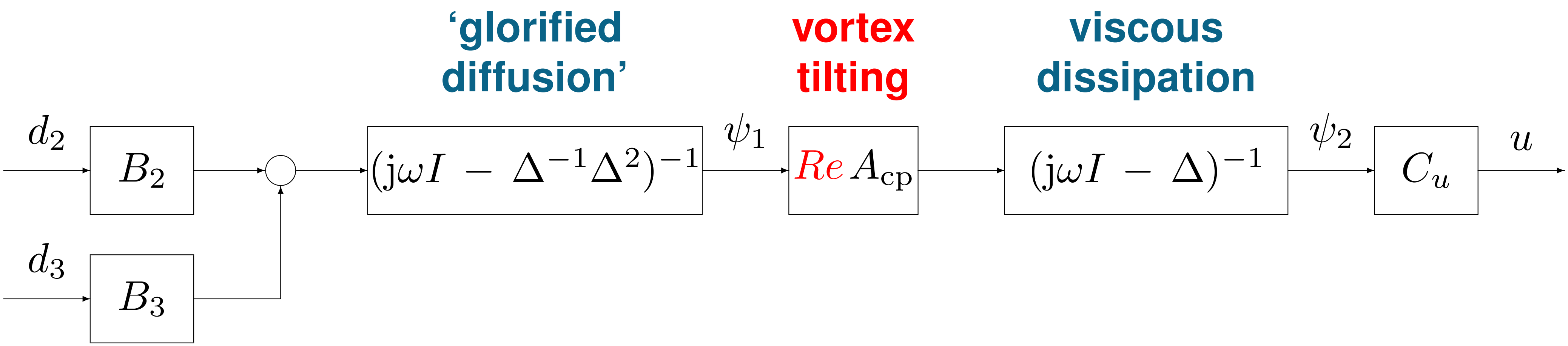
$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 & C_u \\ C_v & 0 \\ C_w & 0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$$



Jovanović & Bamieh, *J. Fluid Mech.* '05

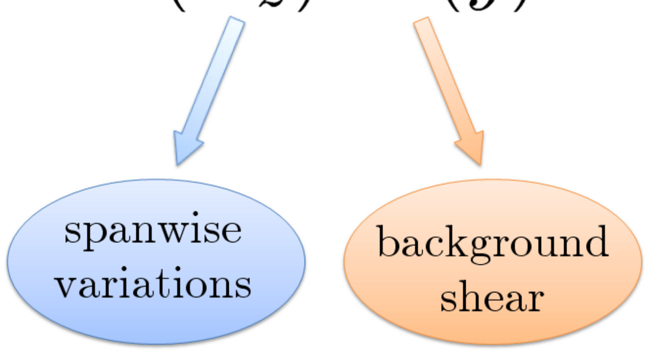
Amplification mechanism in flows with high Re

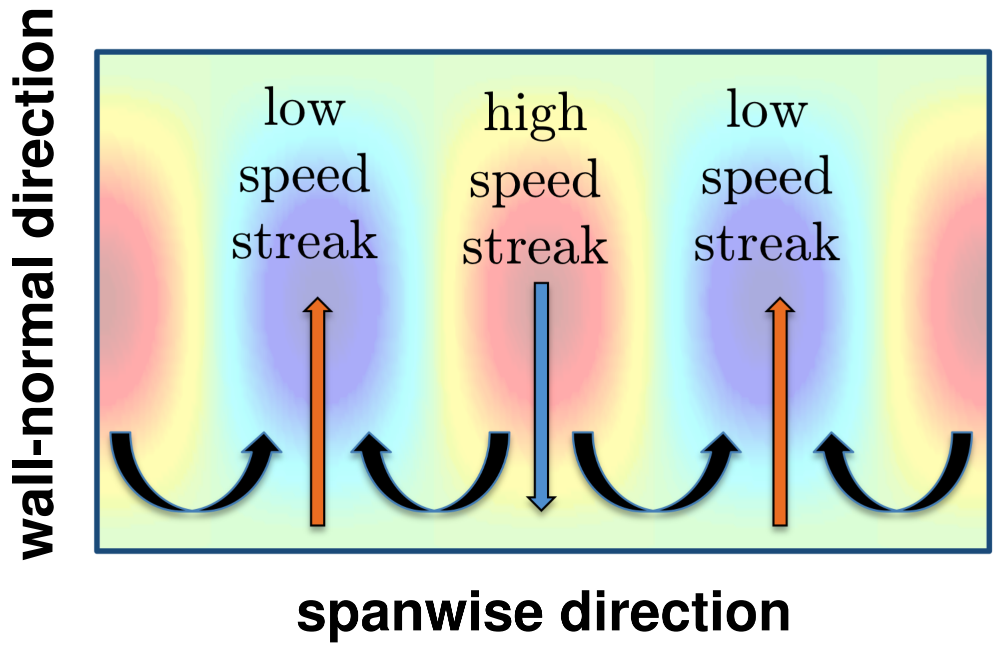
- HIGHEST AMPLIFICATION: $(d_2, d_3) \to u$



☞ AMPLIFICATION MECHANISM: **vortex tilting** or **lift-up**

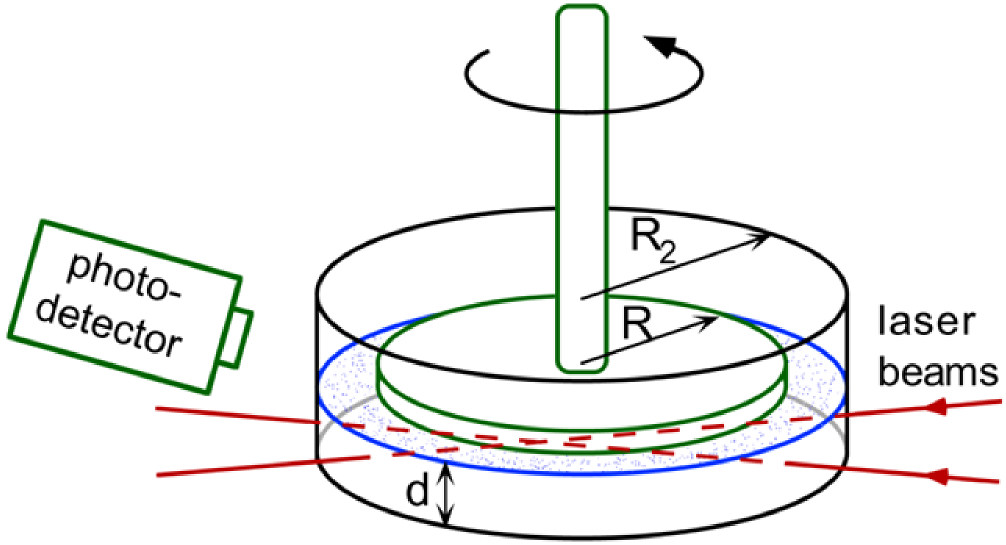
$$A_{cp} = -(\mathrm{i}k_z)\; U'(y)$$





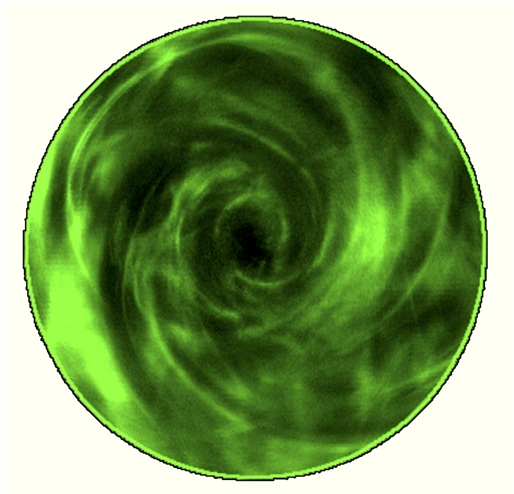
Turbulence without inertia

NEWTONIAN: **inertial turbulence** VISCOELASTIC: **elastic turbulence**



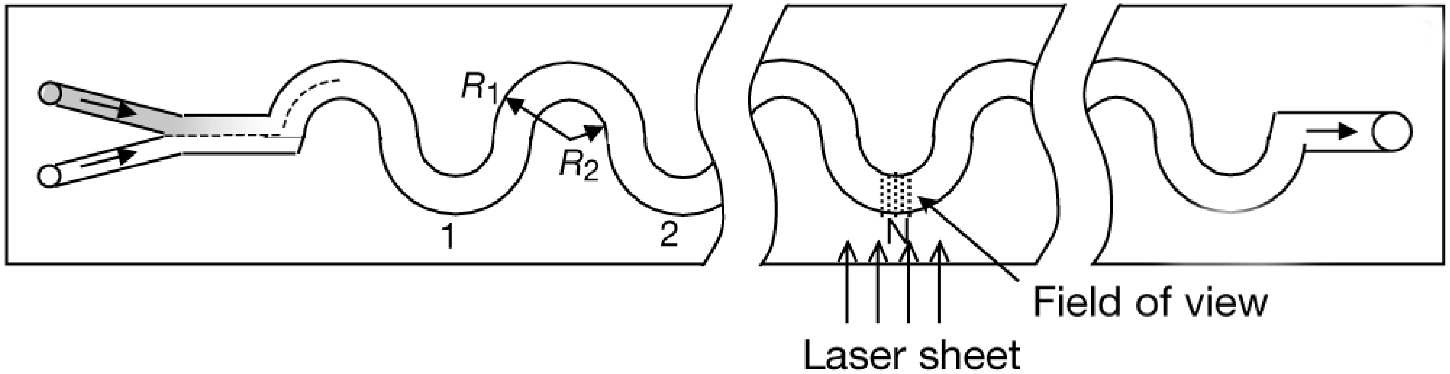
Groisman & Steinberg, *Nature '00*

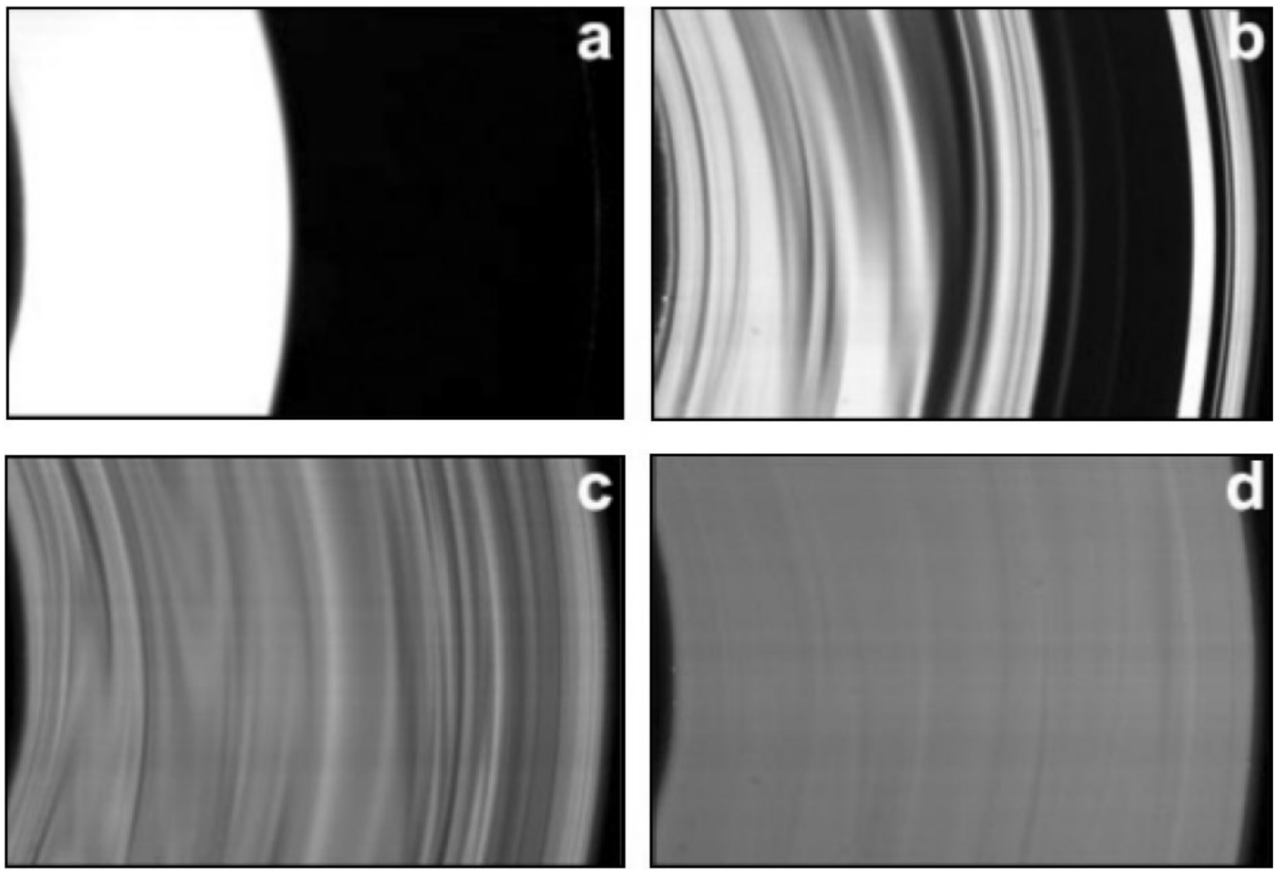
NEWTONIAN: VISCOELASTIC:



☞ FLOW RESISTANCE: **increased 20 times!**

Turbulence: good for mixing . . .

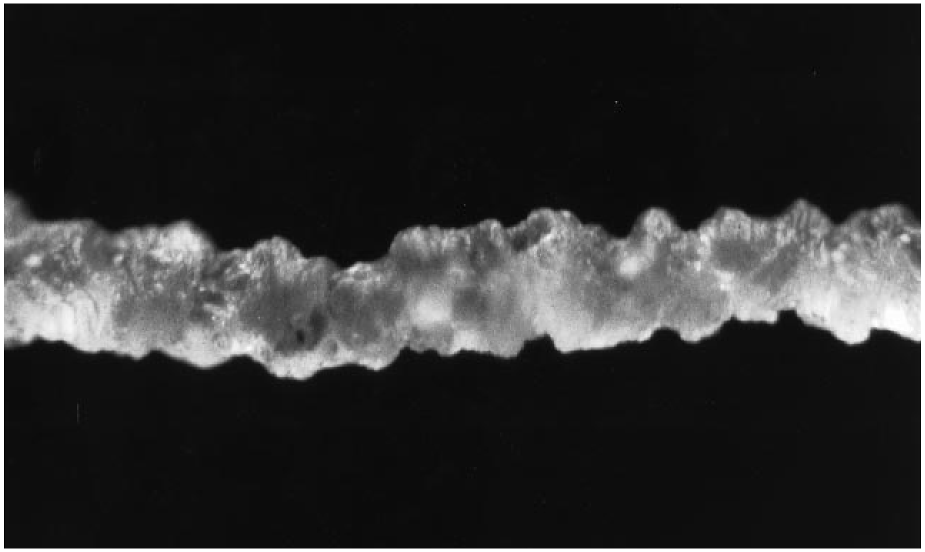




Groisman & Steinberg, *Nature '01*

. . . bad for processing

POLYMER MELT EMERGING FROM A CAPILLARY TUBE

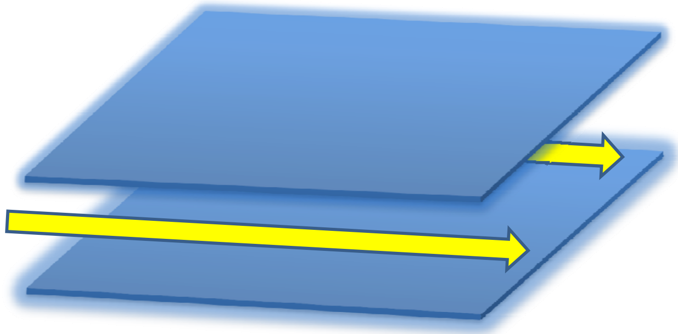


Kalika & Denn, *J. Rheol. '87*

CURVILINEAR FLOWS: **purely elastic instabilities**

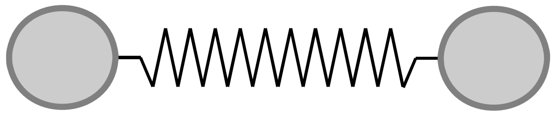
Larson, Shaqfeh, Muller, *J. Fluid Mech. '90*

RECTILINEAR FLOWS: **no modal instabilities**



Oldroyd-B fluids

HOOKEAN SPRING:



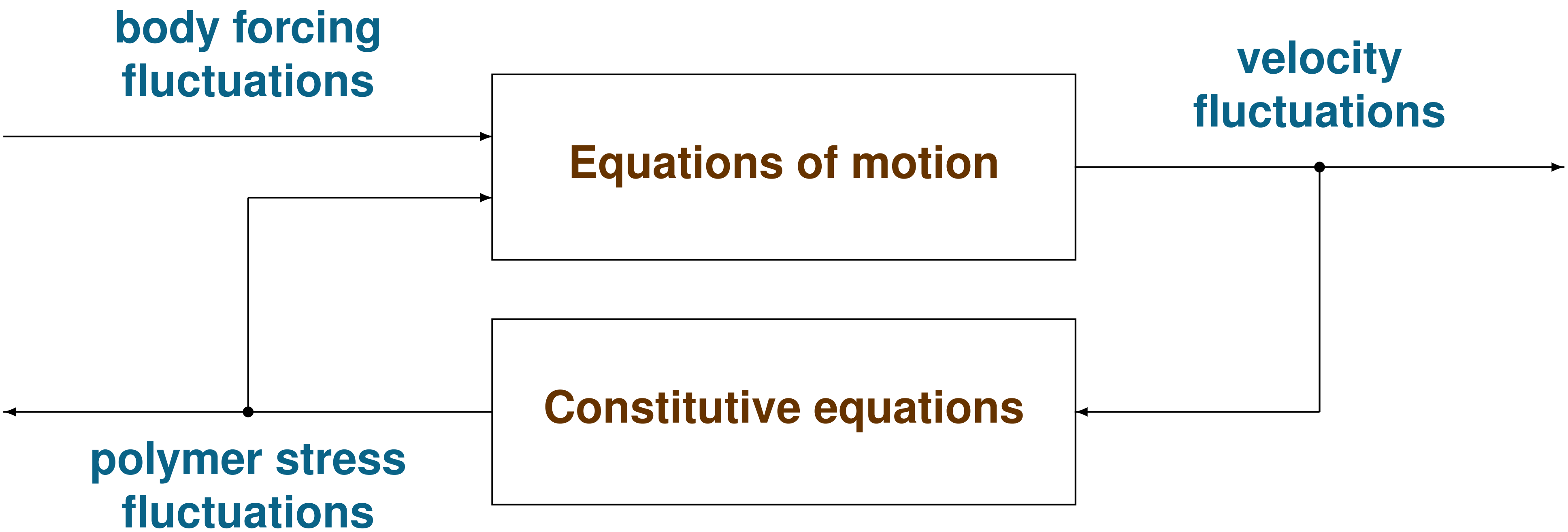
$$
\begin{aligned}
(Re/We)\,\mathbf{v}_t &= -\,Re\,(\mathbf{v}\cdot\nabla)\,\mathbf{v} \;-\; \nabla p \;+\; \beta\,\Delta\mathbf{v} + (1-\beta)\,\nabla\cdot\boldsymbol{\tau} \;+\; \mathbf{d} \\
0 &= \nabla\cdot\mathbf{v} \\
\boldsymbol{\tau}_t &= -\boldsymbol{\tau} \;+\; \nabla\mathbf{v} \;+\; (\nabla\mathbf{v})^T \;+\; We\left(\boldsymbol{\tau}\cdot\nabla\mathbf{v} + (\nabla\mathbf{v})^T\cdot\boldsymbol{\tau} \;-\; (\mathbf{v}\cdot\nabla)\boldsymbol{\tau}\right)
\end{aligned}
$$

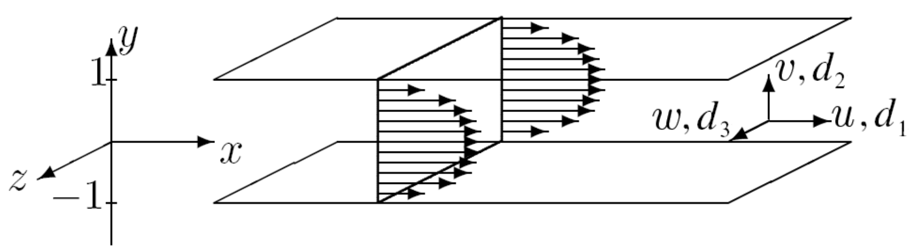
VISCOSITY RATIO: $\beta \;:=\; \dfrac{\text{solvent viscosity}}{\text{total viscosity}}$

WEISSENBERG NUMBER: $We \;:=\; \dfrac{\text{fluid relaxation time}}{\text{characteristic flow time}}$

REYNOLDS NUMBER: $Re \;:=\; \dfrac{\text{inertial forces}}{\text{viscous forces}}$

Input-output analysis





$$\underbrace{\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}}_{\mathbf{d}} \xrightarrow{\text{amplification}} \underbrace{\begin{bmatrix} u \\ v \\ w \end{bmatrix}}_{\mathbf{v}}$$

- INSIGHT INTO AMPLIFICATION MECHANISMS

  importance of streamwise elongated structures

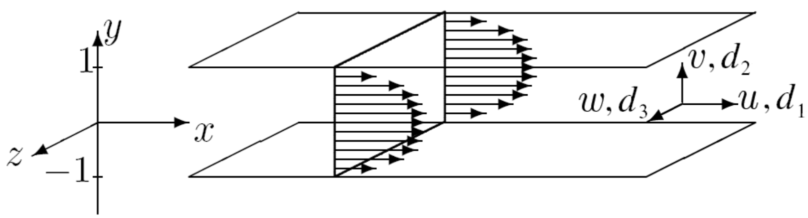
Hoda, Jovanović, Kumar, *J. Fluid Mech. '08, '09*

Jovanović & Kumar, *JNNFM '11*

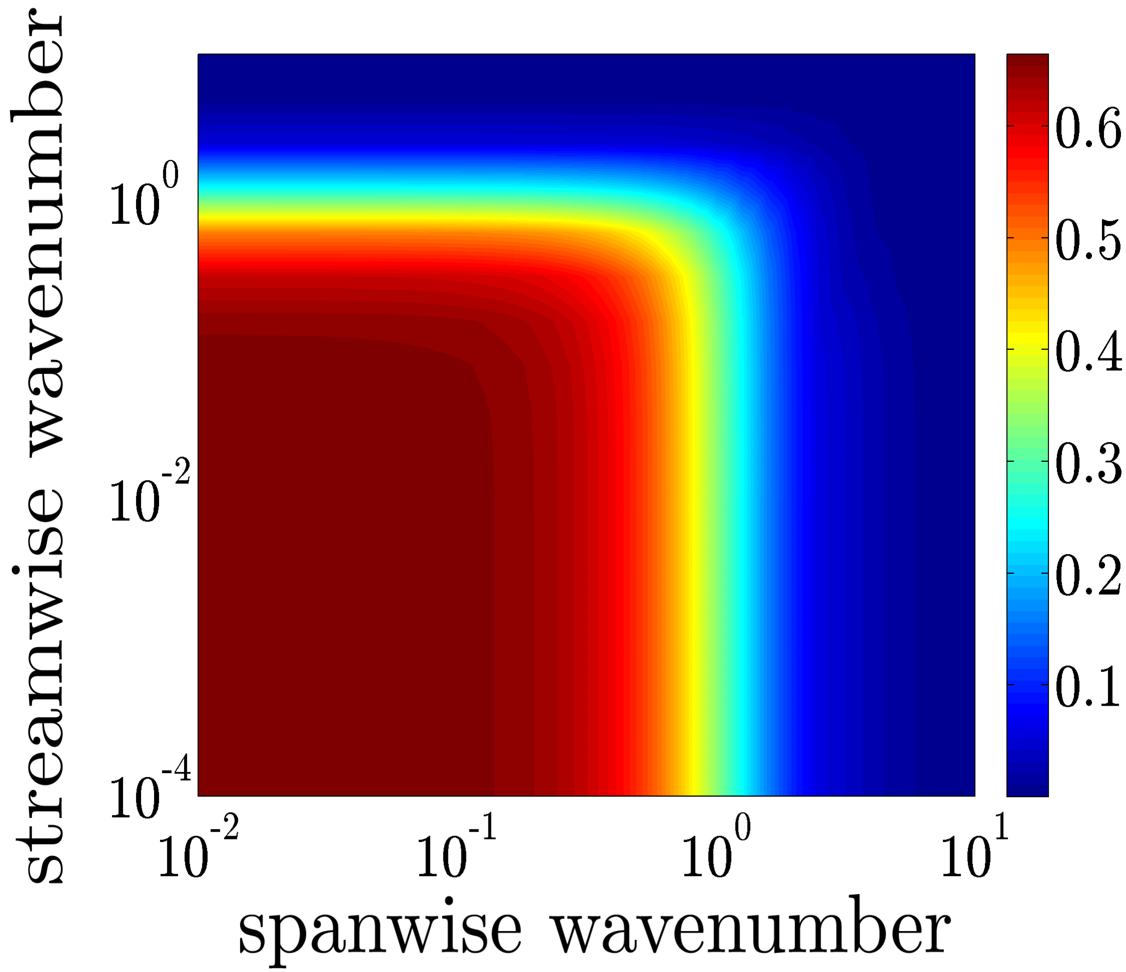
Inertialess channel flow: worst case amplification

- **No single constitutive equation can describe the range of phenomena**
  - ⋆ **important to quantify influence of modeling imperfections on dynamics**

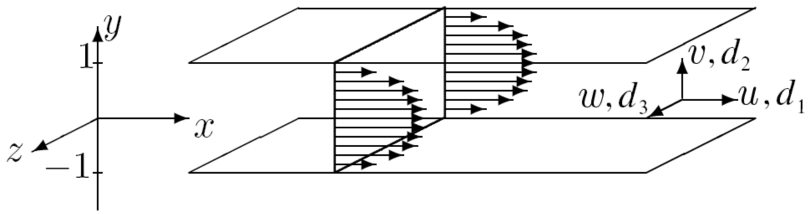
$We = 10$, $\beta = 0.5$, $Re = 0$

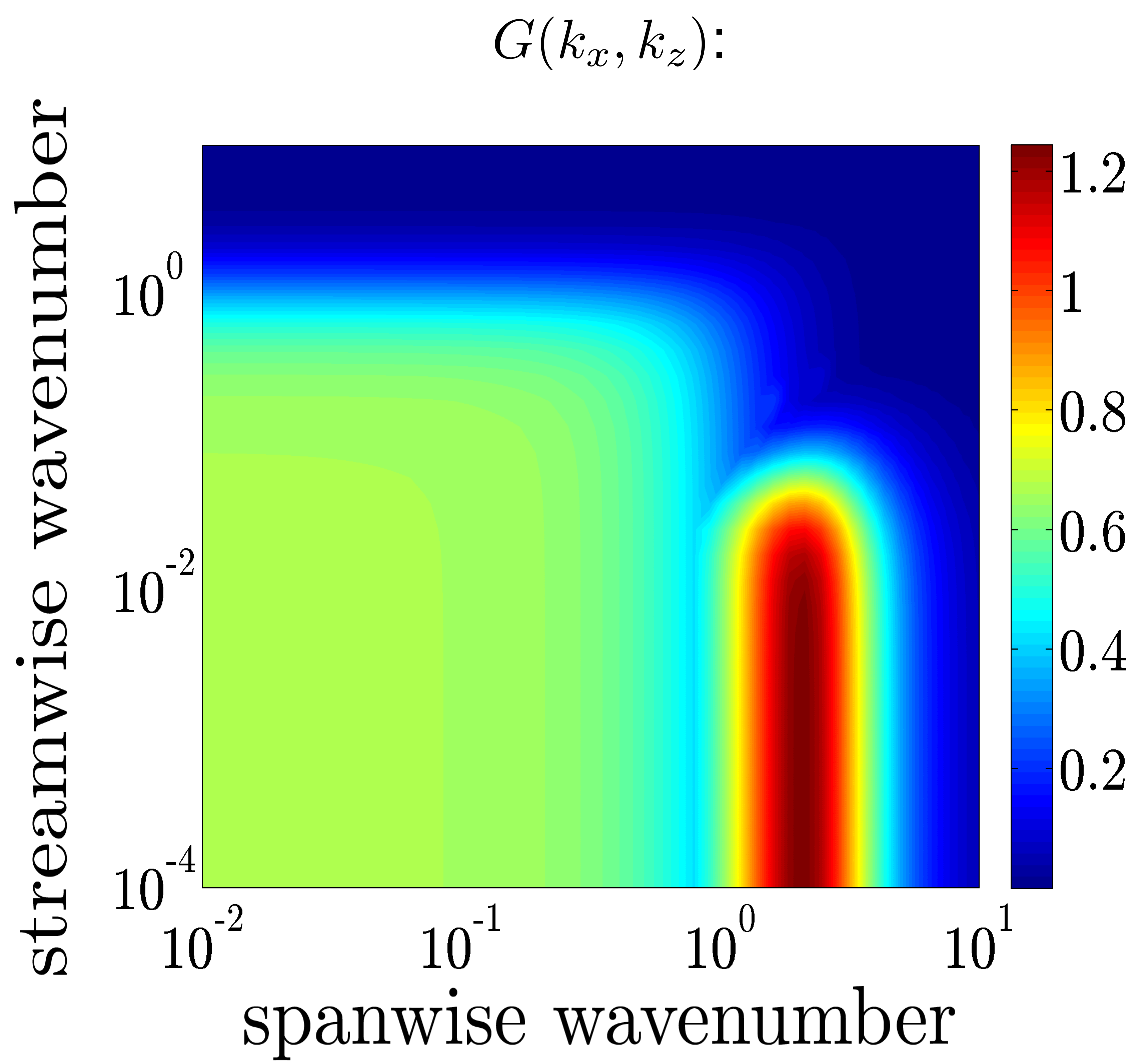


$$G(k_x, k_z) \; = \; \sup_{\omega} \, \sigma^2_{\max} \left(\mathcal{T}(k_x, k_z, \omega) \right):$$

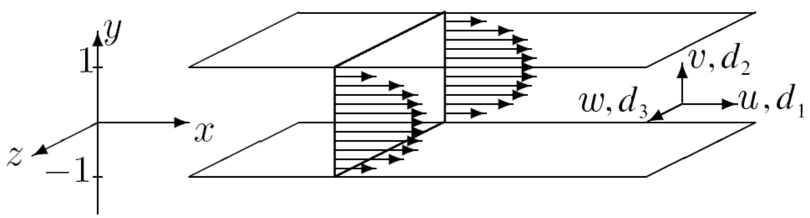


$We = 50$, $\beta = 0.5$, $Re = 0$

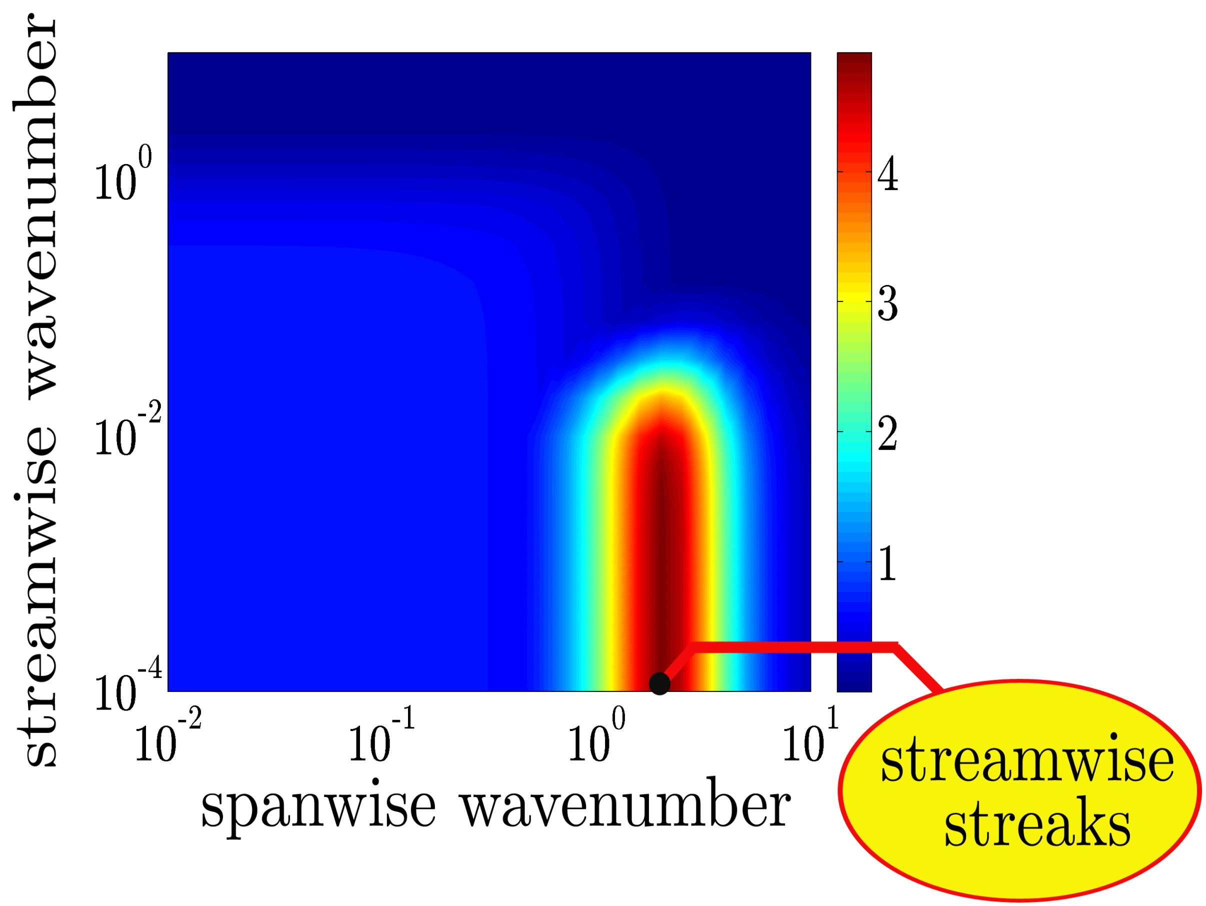




$We = 100$, $\beta = 0.5$, $Re = 0$



$G(k_x, k_z)$:

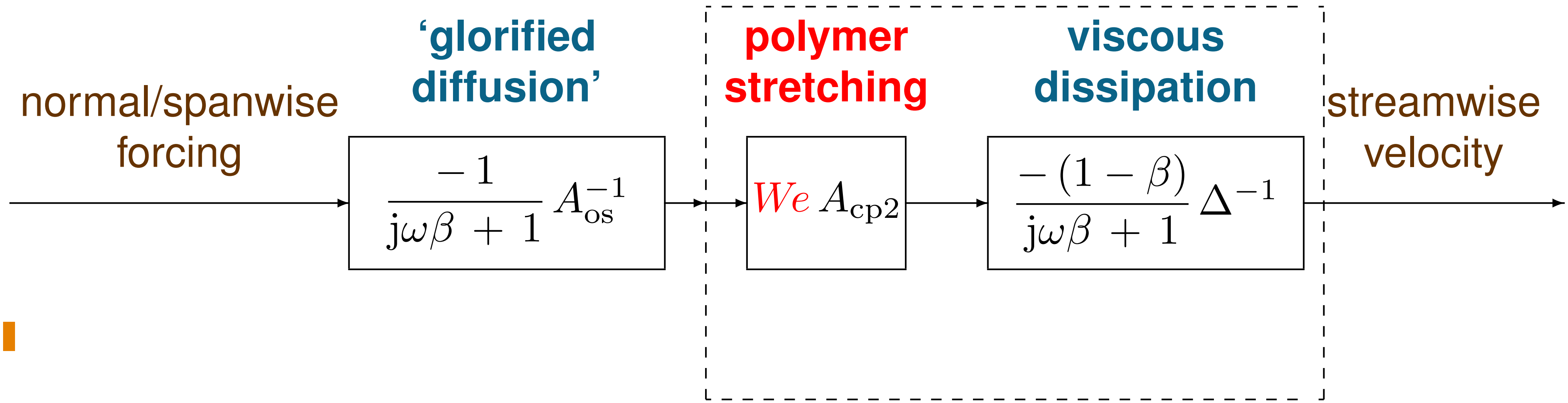


- **Dominance of streamwise elongated structures**
  streamwise streaks!

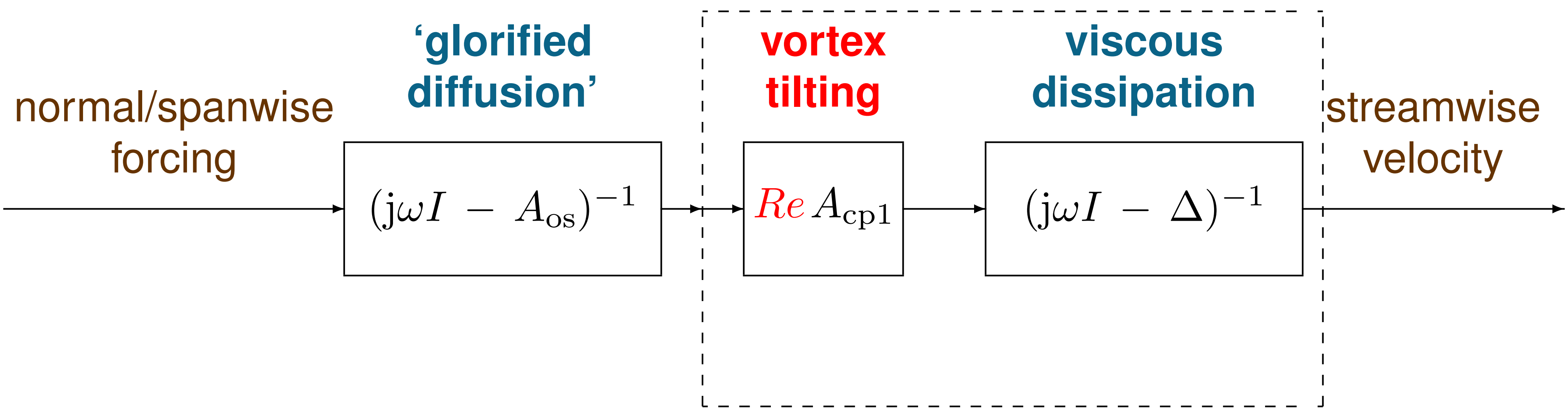
Amplification mechanism

- **Highest amplification:** $(d_2, d_3) \to u$

INERTIALESS VISCOELASTIC:

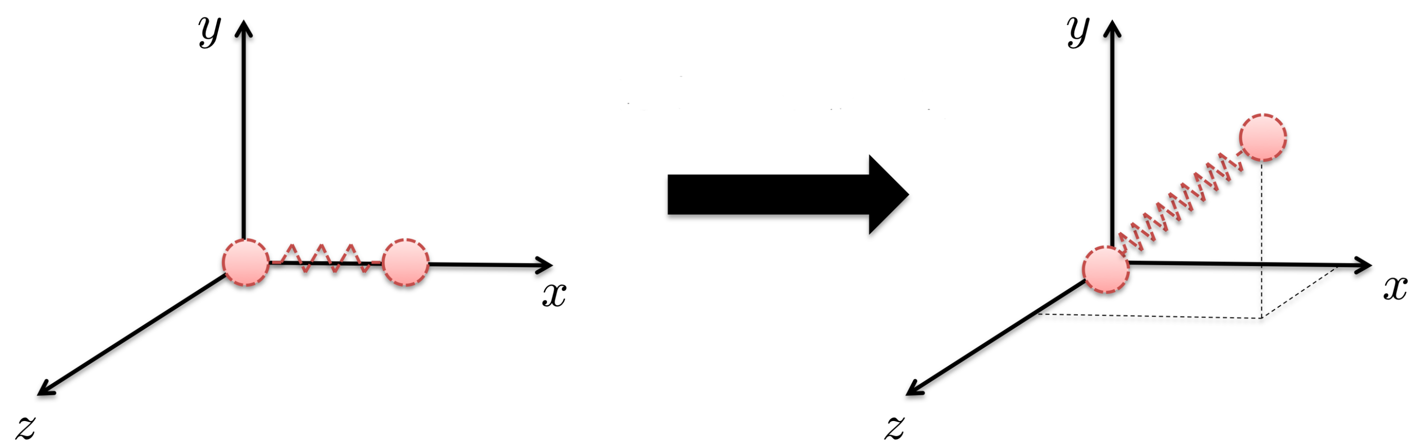


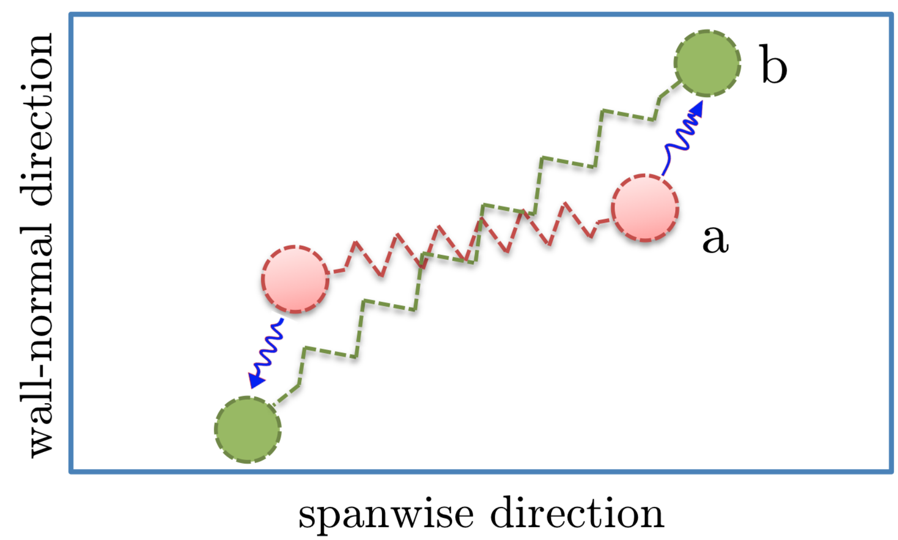
INERTIAL NEWTONIAN:

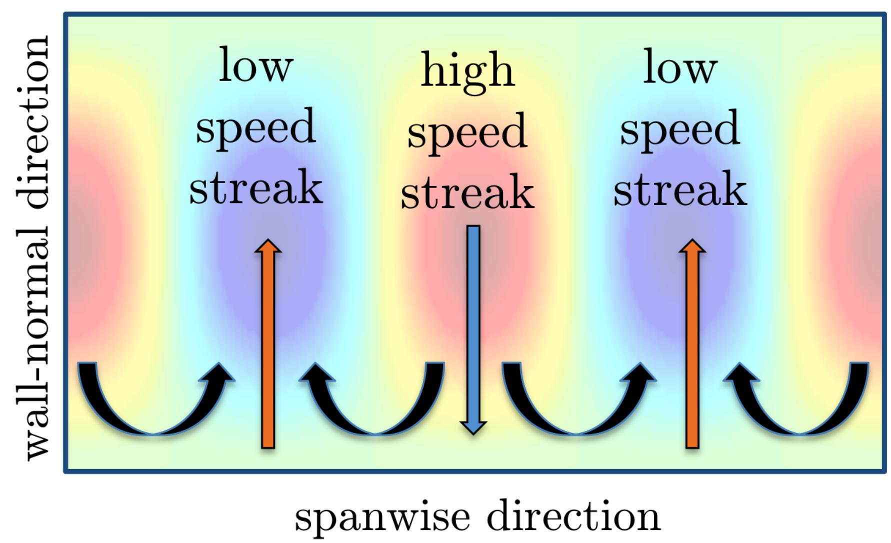


Inertialess lift-up mechanism

$$\begin{aligned}\Delta\,\eta_t &= -(1/\beta)\Delta\eta \;+\; \color{red}{We}\,(1 - 1/\beta)\,A_{\mathrm{cp2}}\,\vartheta \\ &= -(1/\beta)\Delta\eta \;+\; \color{red}{We}\,(1 - 1/\beta)\left(\partial_{yz}\,(U'(y)\,\tau_{22}) + \partial_{zz}\,(U'(y)\,\tau_{23})\right)\end{aligned}$$





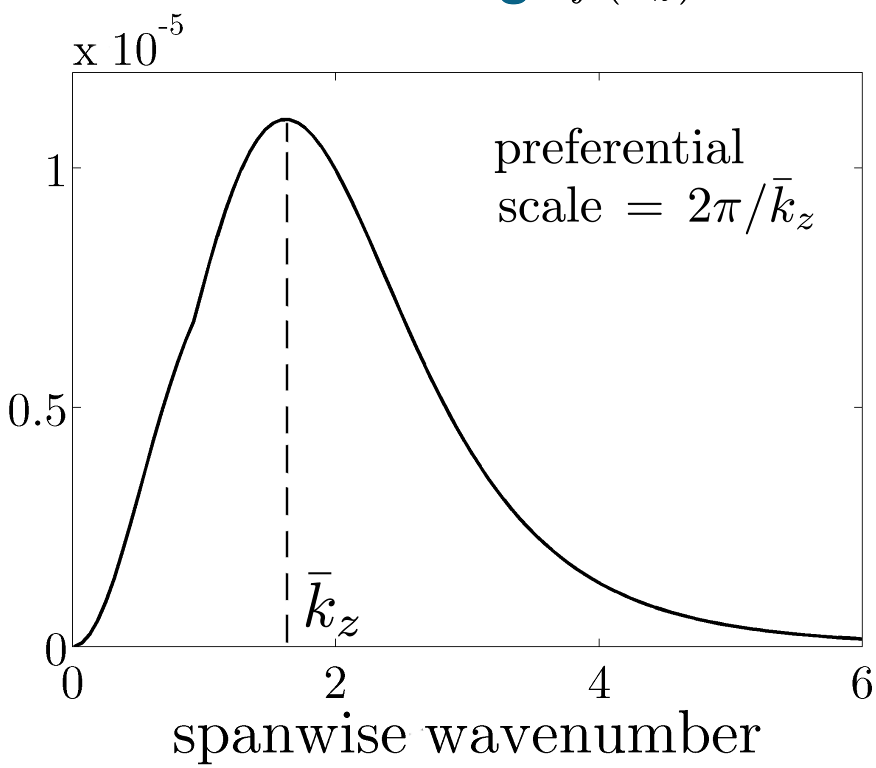


Spatial frequency responses

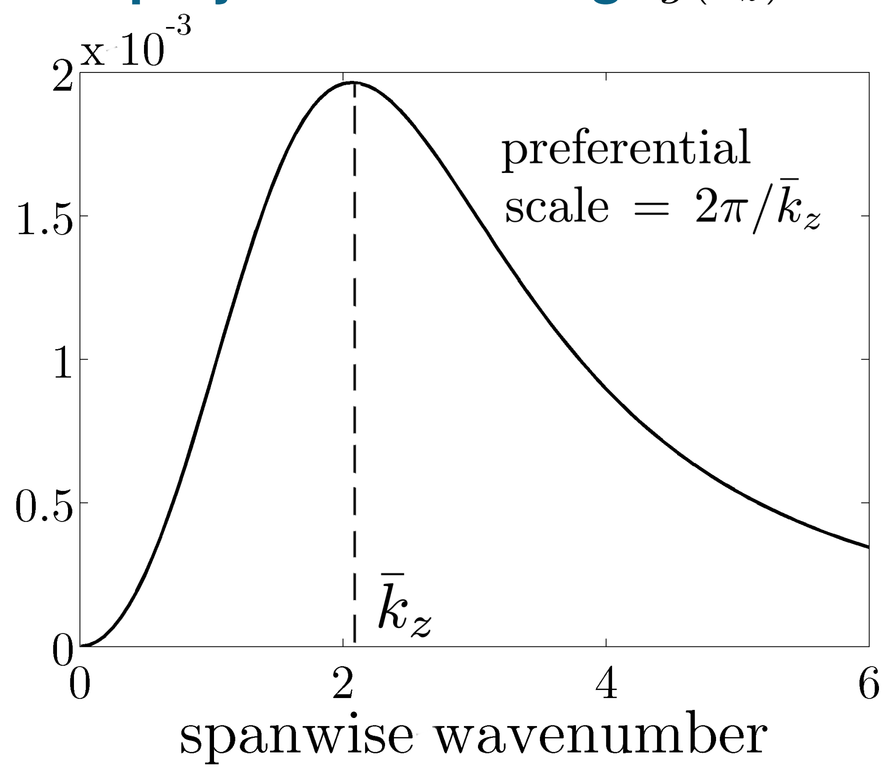
$$(d_2, d_3) \xrightarrow{\text{amplification}} u$$

INERTIAL NEWTONIAN:	$G(k_z; Re) \; = \; Re^2 \, f(k_z)$
INERTIALESS VISCOELASTIC:	$G(k_z; We, \beta) \; = \; We^2 \, g(k_z) \, (1 - \beta)^2$

vortex tilting: $f(k_z)$



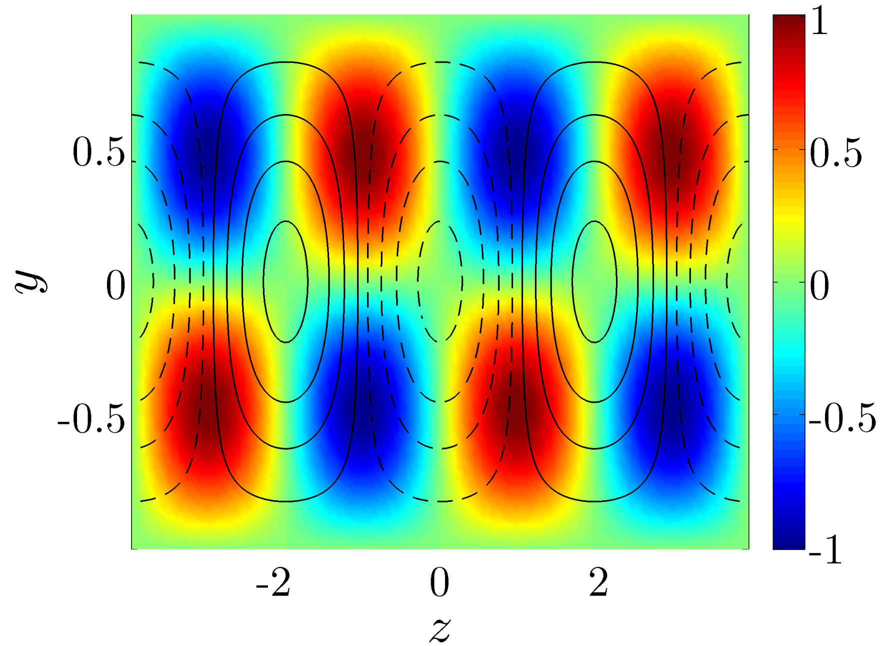
polymer stretching: $g(k_z)$



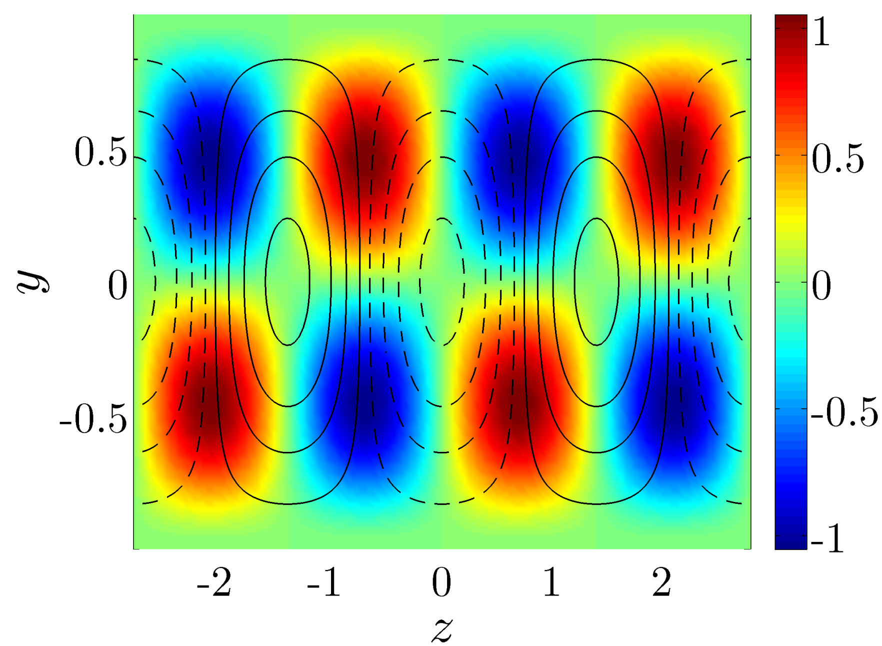
Dominant flow patterns

- **FREQUENCY RESPONSE PEAKS**
  ☞ **streamwise vortices and streaks**

Inertial Newtonian:



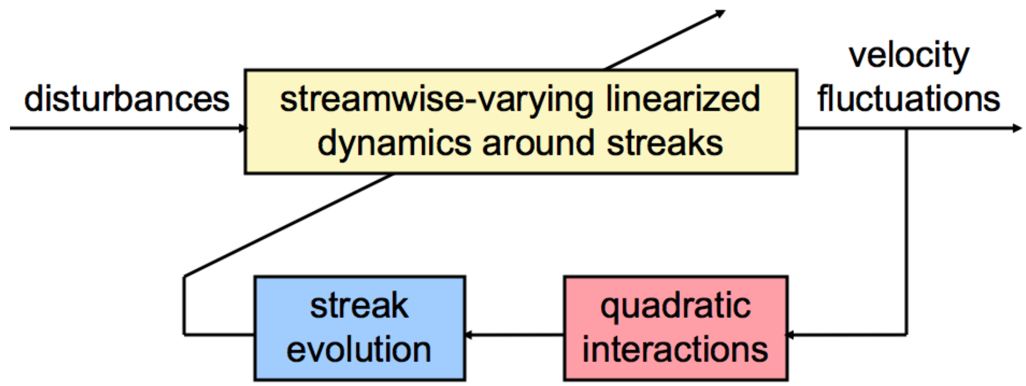
Inertialess viscoelastic:



- **CHANNEL CROSS-SECTION VIEW:**
  - color plots: streamwise velocity
  - contour lines: stream-function

Flow sensitivity vs. nonlinearity

- **Challenge: relative roles of flow sensitivity and nonlinearity**



- **Newtonian fluids: self-sustaining process for transition to turbulence**

  Waleffe, *Phys. Fluids '97*