Due Th 10/04/12 (at the beginning of the class)

1. Use the (matrix) exponential series to evaluate e^{At} for:

   (a) $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$;

   (b) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

2. Suppose $A(t)$ is an $n \times n$ time-varying matrix with continuous entries that satisfies

$$A(t)\left(\int_{t_0}^{t} A(\sigma)\,\mathrm{d}\sigma\right) \;=\; \left(\int_{t_0}^{t} A(\sigma)\,\mathrm{d}\sigma\right) A(t).$$

   Show that the state-transition matrix $\Phi(t_1, t_0)$ can be computed as

$$\Phi(t, t_0) \;=\; \exp\big(\int_{t_0}^{t} A(\sigma)\,\mathrm{d}\sigma\big).$$

3. Find the state transition matrix $\Phi(t_1, t_0)$ for the matrix

$$A(t) \;=\; \begin{bmatrix} \alpha(t) & \beta(t) \\ -\beta(t) & \alpha(t) \end{bmatrix},$$

   where $\alpha(t)$ and $\beta(t)$ are continuous functions of t.

4. (a) Suppose that A and B are constant square matrices. Show that the state transition matrix for the time-varying system

$$\dot{x}(t) \;=\; e^{-At}\, B\, e^{At}\, x(t)$$

   is given by

$$\Phi(t, \tau) \;=\; e^{-At}\, e^{(A+B)(t-\tau)}\, e^{A\tau}.$$

   (b) If A is an $n \times n$ matrix of full rank, use the definition of the matrix exponential to show that that

$$\int_0^t e^{A\sigma}\,\mathrm{d}\sigma \;=\; \left(e^{At} \;-\; I\right) A^{-1}.$$

   Using this result, obtain the solution to the linear time-invariant system

$$\dot{x} = Ax + B\bar{u}\ , \quad x(0) = x_0\ ,$$

   where $\bar{u}$ is a constant vector with m components and B is an $(n \times m)$-dimensional matrix.

5. Consider the nonlinear system

$$\begin{aligned} \dot{x}_1 &= -x_1^2 + x_1 x_2 \\ \dot{x}_2 &= -2x_2^2 + x_2 - x_1 x_2 + 2 \end{aligned}$$

   (a) Show that $\bar{x} = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ is an equilibrium point.

   (b) Is $\bar{x}$ the only equilibrium point?

   (c) Linearize this system around $\bar{x} = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ and find the resolvent and the state-transition matrix of the resulting linearized system.