Due Tuesday 12/04/12

1. Two pendula, coupled by a spring, are to be controlled by two equal and opposite forces u applied to the pendula bobs shown in Figure 1.

The linearized equations of motion are given by

$$ml^2\ddot{\theta}_1 = -ka^2(\theta_1 - \theta_2) - mgl\theta_1 - u,$$

$$ml^2\ddot{\theta}_2 = -ka^2(\theta_2 - \theta_1) - mgl\theta_2 + u.$$

- (a) Determine a state-space representation of the system.
- (b) Is the system controllable? Explain mathematically and physically.
- (c) Determine the range space of the controllability matrix.Hint: use singular value decomposition (Matlab's command svd).
- (d) Determine the transfer function from u to θ_1 .
- (e) Assume m = 1, l = 1, a = 0.5, k = 4, g = 10. Is the system stable?



Figure 1: Two pendula coupled by a spring.

2. Consider the following pair of matrices

$$A = \begin{bmatrix} \lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3 \end{bmatrix}, \quad B = \begin{bmatrix} \gamma_1\\ \gamma_2\\ \gamma_3 \end{bmatrix}.$$

- (a) Suppose λ_1 , λ_2 , and λ_3 are distinct, what are the conditions on γ_1 , γ_2 , and γ_3 for the pair (A, B) to be controllable?
- (b) For a discrete-time system with the above given matrices A and B, $\{\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3\}$, and $\gamma_1 = \gamma_2 = \gamma_3 = 1$, plot the energy (as a function of time for $k \ge 3$) of the minimum energy control necessary to reach the final state $x_f = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$. Comment your results.
- (c) Suppose λ_1 , λ_2 , and λ_3 are not necessarily distinct, how does the above condition change?
- (d) Generalize the results that you obtained for the above two cases to the pair (A, B) where

$$A = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}, \quad B = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{bmatrix}.$$

3. Consider the following matrices

$$A_{1} = \begin{bmatrix} \lambda_{1} & 1 & 0 \\ 0 & \lambda_{1} & 1 \\ 0 & 0 & \lambda_{1} \end{bmatrix}, \quad B_{1} = \begin{bmatrix} \gamma_{1} \\ \gamma_{2} \\ \gamma_{3} \end{bmatrix},$$
$$A_{2} = \begin{bmatrix} \lambda_{2} & 1 & 0 \\ 0 & \lambda_{2} & 1 \\ 0 & 0 & \lambda_{2} \end{bmatrix}, \quad B_{2} = \begin{bmatrix} \beta_{1} \\ \beta_{2} \\ \beta_{3} \end{bmatrix}.$$

- (a) What are the conditions on γ_1 , γ_2 and γ_3 for the pair (A_1, B_1) to be reachable?
- (b) Using the matrices A_1 , A_2 , B_1 and B_2 we can construct the following pair of matrices

$$A = \left[\begin{array}{cc} A_1 & 0 \\ 0 & A_2 \end{array} \right], \quad B = \left[\begin{array}{c} B_1 \\ B_2 \end{array} \right].$$

- i. Suppose λ_1 and λ_2 are distinct, what are the conditions on γ_1 , γ_2 , γ_3 , β_1 , β_2 , and β_3 for the pair (A, B) to be reachable?
- ii. Suppose $\lambda_1 = \lambda_2$, what are the conditions on γ_1 , γ_2 , γ_3 , β_1 , β_2 , and β_3 for the pair (A, B) to be reachable?
- 4. Suppose A is $n \times n$ matrix, and B is $n \times m$ matrix. Prove that (A, B) is controllable if and only if (A + BK, B) is controllable for all $m \times n$ matrices K.
- 5. Let $\dot{x} = Ax + Bu$ be a single-input linear system with state dimension $n \ge 2$. Show that if B is an eigenvector of A then the system is not reachable.
- 6. Given the linear time-invariant system

$$\dot{x} = \begin{bmatrix} -7 & -2 & 6\\ 2 & -3 & -2\\ -2 & -2 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 1\\ 1 & -1\\ 1 & 0 \end{bmatrix} u =: Ax + Bu$$
$$y = \begin{bmatrix} -1 & -1 & 2\\ 1 & 1 & -1 \end{bmatrix} =: Cx,$$

- (a) Check controllability using:
 - The controllability matrix (the Kalman rank test).
 - Rows of $\hat{B} := Q^{-1}B$, where Q is chosen such that $Q^{-1}AQ$ is diagonal.
 - The PBH test.
- (b) Identify the controllable and uncontrollable modes of the system, and convert the system to *a* Kalman controllable canonical form.