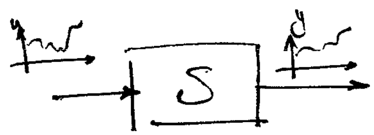
Linear systems 09/06/12

## Last time



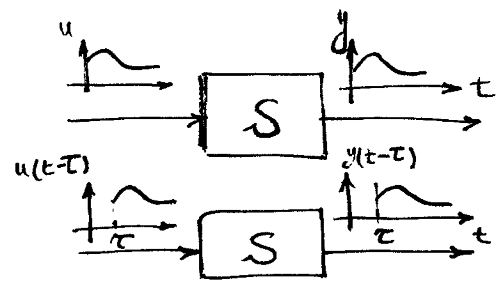

$S$ : system (mapping from inputs to outputs)

1. linearity

today
2. time invariance
3. causality
4. memory

state-space representation

* Time invariance




Shift operator

$$[\sigma_\tau u](t) = u(t-\tau)$$

Now the system is time invariant if

$$[S\sigma_\tau u](t) = [\sigma_\tau S u](t)$$

Shifted input:

$$\bar{u}(t) = [\sigma_\tau u](t) = u(t-\tau)$$

$$\bar{y}(t) = [S\bar{u}](t) \overset{\text{if we have this!}}{=} y(t-\tau)$$

where $\quad y(t) = [Su](t)$

Abstractly: System is time invariant

$S$ and $\sigma_\tau$ must commute:

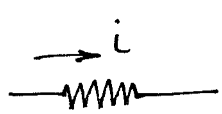
$$S\sigma_\tau = \sigma_\tau S$$

Inverse of shift operator: $\sigma_{-\tau} = \sigma_\tau^{-1}$

$S$ is time invariant $:\longrightarrow$ $S = \sigma_{-\tau} S \sigma_\tau$

**Ex. 1**

Resistor




(a) $V(t) = R \cdot i(t)$

$y(t) = R \cdot u(t)$

$i$: input

$V$: output

$R$: const

(b) $V(t) = R(t)\, i(t)$

$y(t) = \underbrace{R(t)}_{\text{e.g. } t} \cdot i(t)$

→ how can we show that system is time invariant or not ?

for case (a) $\quad \bar{u}(t) = u(t-\tau)$

$$\bar{y}(t) = R u(t-\tau)$$
$$= y(t-\tau) \checkmark$$

for case (b) $\quad y(t) = R(t)\, u(t)$

$$y(t-\tau) = R(t-\tau)\, u(t-\tau)$$

if we shif $u(t) \to u(t-\tau)$ we get $R(t)\,u(t-\tau)$ which is not equal to $y(t-\tau)$! for this system to be time invariant $R(t)$ had to shift as well. case (b) is not time invariant.

Ex.2 $\quad y(t) = \int_{-\infty}^{t} H(t,\tau)\, u(\tau)\, d\tau$

→ you will get a chance to practice this in your HW

fact: time-invariant

$$\Updownarrow$$

$$H(t,\tau) = H(t-\tau)$$

## * Causality

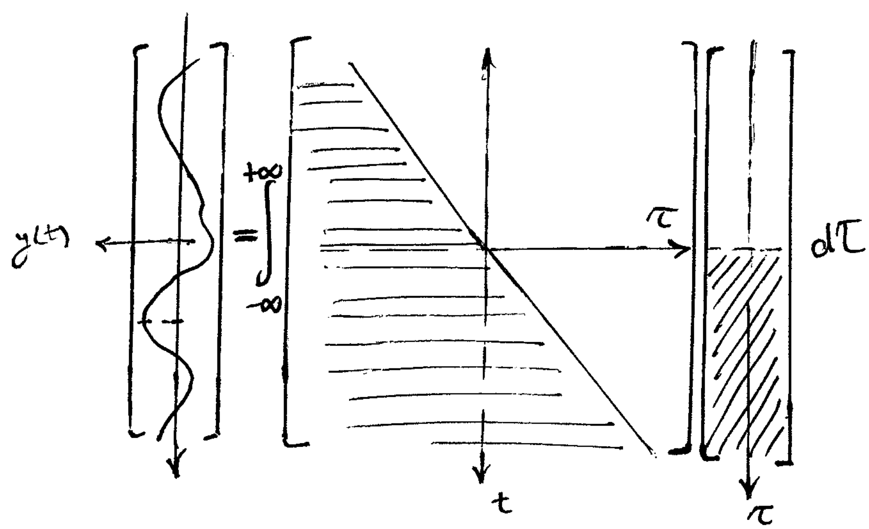
System $S$ is causal

$$\Updownarrow$$

current output doesn't depend on future inputs

$$y(t) = \int_{-\infty}^{+\infty} H(t-\tau)\cdot u(\tau)\,d\tau = \int_{-\infty}^{\infty} H(\tau)\cdot u(t-\tau)\,d\tau$$

current time




$$H(t,\tau) = \{0,\ t-\tau < 0\}$$

$H(t-\tau)$

## * Memoryless

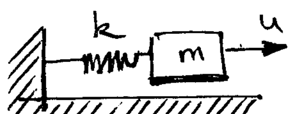
System is static or memoryless if current output only depends on current input.

Ex. Resistor, saturation, dead-zone, lever

$\hookrightarrow$ memoryless or static systems

Ex. dynamic systems:

$$y(t) = \int_{-\infty}^{+\infty} H(t-\tau)u(\tau)\,d\tau$$




$$m\cdot\frac{d^2y}{dt} + ky = u$$

$$y(T) = f\left[u[0,T]\,;\, y(0),\, \dot{y}(0)\right]$$

for the output at T, we need to know the entire history of our system, up to that time

$$y(t) = \int H(t-\tau)\cdot u(\tau)\,d\tau = k\,u(t)$$

$$\downarrow$$

$$H(t-\tau) = k\,\delta(t-\tau)$$ $\longrightarrow$ the delta function

$$y(t) = k\int_{-\infty}^{+\infty}\delta(t-\tau)\,u(\tau)\,d\tau = k\,u(t)$$

system is memoryless if its impulse response is a delta function

# State-space models :

state equation : $\frac{dx}{dt} = f(x,u,t)$ (1)

output equation : $y = g(x,u,t)$ (2)

$t$: time
$y$: output
$u$: input
$x$: state

(1) 1st order diff. equations in time
(vector valued)

(2) static in time equation

Ex the equation

$$m.\frac{d^2y}{dt^2} + ky = u$$

is not in the state space form!

choose (a) $x_1 = y \longrightarrow \dot{x}_1 = \dot{y} \overset{(b)}{=} x_2$
(b) $x_2 = \dot{y} \longrightarrow \dot{x}_2 = \ddot{y} = -\frac{k}{m}x_1 + \frac{1}{m}u$

$\dot{x} = \frac{dx}{dt}$

$\ddot{y} = \frac{d^2y}{dt^2}$

why do we not need $x_3 = \ddot{y}$ ?
do we need this information?

$$(I) \Rightarrow \quad \ddot{y} = -\frac{k}{m} y + \frac{1}{m} u \overset{(a)}{=} -\frac{k}{m} x_1 + \frac{1}{m} u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{k}{m} x_1 + \frac{1}{m} u \end{bmatrix} \qquad \text{How} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\dot{x} = f(x, u) = \begin{bmatrix} x_2 \\ -\frac{k}{m} x_1 + \frac{1}{m} u \end{bmatrix} = \begin{bmatrix} f_1(x, u) \\ f_2(x, u) \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]\, u$$

$$\dot{x}_1 = 0.x_1 + 1.x_2 + 0.u$$

$$\dot{x}_2 = -\frac{k}{m} x_1 + 0.x_2 + \frac{1}{m} u \quad \rightarrow \quad \text{fill the matrices!}$$

$$y = 1.x_1 + 0.x_2 + 0.u$$

In a general case when: $f(x,u,t) = A(t)x(t) + B(t)u(t)$

$g(x,u,t) = C(t)x(t) + D(t)u(t)$

we'll end up with a linear state-space model.

For systems with $m$ inputs
$p$ outputs
$n$ states

at any time instant the coorresponding input, output and states (vectors with $m$, $p$ and $n$ elements)

$$u(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{bmatrix} \in \mathbb{R}^m \qquad x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \in \mathbb{R}^n$$

$$y(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_p(t) \end{bmatrix} \in \mathbb{R}^p$$

↳ in our mass-spring example $m=1$
$p=1$
$n=2$

Note! Whenever you have input-output differential equation of the form: $F(y^{(N)}, \dots, y, u) = 0$

where there are no derivatives of input w.r.t time, then the states can be selected as physical variables (output and its corresponding derivatives)

$x_1 = y$
$x_2 = \dot{y}$
$\vdots$
$x_n = y^{(N-1)}$