Linear Systems 09/11/12

Last time:

state space models
$$\begin{cases} \dot{x}(t) = f(x(t), u(t), t) & (1) \\ y(t) = g(x(t), u(t), t) & (2) \end{cases}$$

Today: continuation

(1): state equation (dynamics)

(2): output equation (static in time eq.)

$$\dot{x}(t) = \frac{dx(t)}{dt}$$

For MS system $\quad x(t) = \begin{pmatrix} \text{position} \\ \text{velocity} \end{pmatrix}$

mass spring

We will focus on non-linear time-invariant systems:

functions f & g don't depend explicitly on time

$$\dot{x} = f(x, u) \quad (1)$$
$$y = g(x, u) \quad (2)$$

Question: Given a pair $(\bar{x}(t), \bar{u}(t))$ under what condition is this pair a solution to (1) ?

A pair $(\bar{x}(t), \bar{u}(t))$ is a trajectory of system (1) if

$$\begin{cases} \dot{\bar{x}}(t) = f(\bar{x}(t), \bar{u}(t)) \\ \bar{x}(0) = x_0 \qquad \text{(initial condition)} \end{cases}$$

In particular we'll pay attention to trajectories that don't change with time:

$$(\bar{x}(t), \bar{u}(t)) = (\bar{x}, \bar{u}) = \text{const.}$$

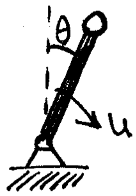
Since $\bar{x} = \text{const} \Rightarrow \dot{\bar{x}} = \frac{d\bar{x}}{dt} = 0 \Rightarrow \boxed{0 = f(\bar{x}, \bar{u})} \quad (*)$

Any $(\bar{x}, \bar{u})$ that satisfy $(*)$ are called equilibrium points of system (1).

→ We'll pay particular attention to equilibrium points with $\bar{u} = 0$. In other words $0 = f(\bar{x}, 0)$

No input ⇒ eq. points determine points s.t. "if you start @ them you always stay there"

Ex. 1 Inverted pendulum



$$\ddot{\theta} - \sin\theta = u$$

θ : angular position (angle)

$\dot{\theta}$: angular velocity

$\ddot{\theta}$: angular acceleration

u : torque (input)

2 states

no dots on the top of input $\Rightarrow$ can choose physical states ✓

$x_1 = \theta$ (position)

$x_2 = \dot{\theta}$ (velocity)

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ \sin(x_1) + u \end{pmatrix}$$

$$y = x_1$$

$$f(x,u) = \begin{pmatrix} f_1(x,u) \\ f_2(x,u) \end{pmatrix} = \begin{pmatrix} x_2 \\ \sin(x_1) + u \end{pmatrix}$$

$$g(x,u) = x_1$$

lets find the equilibria

set $\bar{u} = 0$ and determine solution to $f(\bar{x}, 0) = 0$

$$\begin{pmatrix} x_2 \\ \sin(x_1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\bar{x}_2 = 0$

$\sin(\bar{x}_1) = 0 \quad \Rightarrow \bar{x}_1 = K\pi \quad k = 0, \pm 1, \pm 2, \ldots$

$$\bar{x}_{up} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \bar{x}_{1,up} \\ \bar{x}_{2,up} \end{pmatrix} \; ; \; \bar{x}_{down} = \begin{pmatrix} \pi \\ 0 \end{pmatrix} = \begin{pmatrix} \bar{x}_{1,down} \\ \bar{x}_{2,down} \end{pmatrix}$$

Question: Study the behavior of fluctuations around system trajectories.

$\dot{\bar{x}}(t) = f(\bar{x}(t), \bar{u}(t))$

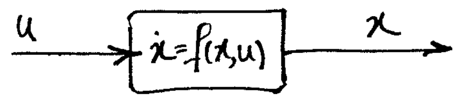
Represent $x(t)$ and $u(t)$ as :

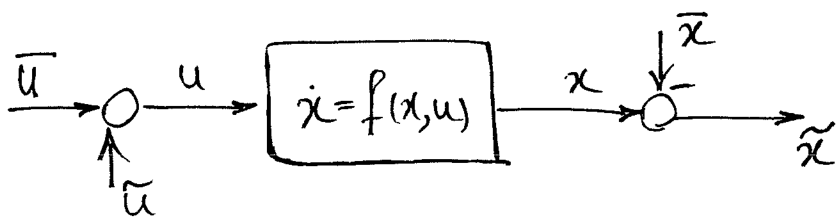
$x(t) = \bar{x}(t) + \tilde{x}(t) \qquad (3)$

trajectory

deviation (fluctuation, perturbation) from $\bar{x}(t)$

and similarly for $u(t)$: $\qquad u(t) = \bar{u}(t) + \tilde{u}(t) \quad (4)$





We are interested in studying the behavior of $\tilde{u}$ and $\tilde{x}$.

sub. in

$(3) \,\&\, (4) \longrightarrow (1)$

$$\cancel{\dot{\bar{x}}(t)} + \dot{\tilde{x}}(t) = f(\bar{x}+\tilde{x}, \bar{u}+\tilde{u})$$

$$\bar{y}(t) + \tilde{y}(t) = g(\bar{x}+\tilde{x}, \bar{u}+\tilde{u})$$

we had $f(\bar{x},\bar{u})$ we want equation for $\tilde{x}$:

$$\dot{\tilde{x}} = f(\bar{x}+\tilde{x}, \bar{u}+\tilde{u}) - f(\bar{x},\bar{u}) \rightarrow \dot{\bar{x}} \qquad (\mathrm{I})$$

$$\tilde{y} = g(\bar{x}+\tilde{x}, \bar{u}+\tilde{u}) - g(\bar{x},\bar{u}) \rightarrow \bar{y} \qquad (\mathrm{II})$$

Note! No approximations have been made.

Now we'll assume that $\tilde{x}, \tilde{u}, \tilde{y}$ are small (in some sense)

We can use Taylor series to represent 1st terms on the RHS of (I) and (II) as

$(\cancel{\bar{x}} + \tilde{x} - \cancel{\bar{x}})$

$$f(\bar{x}+\tilde{x}, \bar{u}+\tilde{u}) = f(\bar{x},\bar{u}) + \left.\frac{\partial f}{\partial x}\right|_{(\bar{x},\bar{u})} \cdot \tilde{x} + \left.\frac{\partial f}{\partial u}\right|_{(\bar{x},\bar{u})} \cdot \tilde{u} + \text{H.O.T.}$$

the same can be written for g.

H.O.T. : Terms of $\|\tilde{x}\|^2, \|\tilde{u}\|^2$ or higher order

$\|\cdot\|$: a norm of a vector

→ By doing this we will acheive the following form:

$$\dot{\tilde{x}} = \left.\frac{\partial f}{\partial x}\right|_{(\bar{x},\bar{u})} \cdot \tilde{x} + \left.\frac{\partial f}{\partial u}\right|_{(\bar{x},\bar{u})} \cdot \tilde{u}$$

$$\tilde{y} = \left.\frac{\partial g}{\partial x}\right|_{(\bar{x},\bar{u})} \cdot \tilde{x} + \left.\frac{\partial g}{\partial u}\right|_{(\bar{x},\bar{u})} \cdot \tilde{u}$$

$\left.\frac{\partial f}{\partial x}\right|_{\bar{x},\bar{u}}$: Jacobian of function f w.r.t. x evaluated @ $(\bar{x},\bar{u})$ and similarly for others

Back to I.P. Ex.

$$f(x,u) = \begin{bmatrix} x_2 \\ \sin(x_1)+u \end{bmatrix} = \begin{bmatrix} f_1(x_1,x_2,u) \\ f_2(x_1,x_2,u) \end{bmatrix}$$

$g(x,u)$

$$\frac{\partial f}{\partial x} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \cos(x_1) & 0 \end{pmatrix}$$

$$\frac{\partial f}{\partial u} = \begin{pmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\frac{\partial g}{\partial x} = \begin{pmatrix} \frac{\partial g}{\partial x_1} , \frac{\partial g}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$\frac{\partial g}{\partial u} = 0$$

$\rightarrow$ pay attention to the dimensions of these Jacobian matrices.

$$A = \frac{\partial f}{\partial x}\Big|_{(\bar{x},\bar{u})} \quad ; \quad B = \frac{\partial f}{\partial u}\Big|_{(\bar{x},\bar{u})}$$

$$C = \frac{\partial g}{\partial x}\Big|_{(\bar{x},\bar{u})} \quad ; \quad D = \frac{\partial g}{\partial u}\Big|_{(\bar{x},\bar{u})}$$

For : (a) upper equilibrium $\left(\begin{pmatrix}0\\0\end{pmatrix}, 0\right) = (\bar{x}_{up}, \bar{u}_{up})$

$$A_{up} = \begin{pmatrix} 0 & 1 \\ \cos(0) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

For : (b) lower equilibrium $\left(\begin{pmatrix}\pi\\0\end{pmatrix}, 0\right) = (\bar{x}_{down}, \bar{u}_{down})$

$$A_{down} = \begin{pmatrix} 0 & 1 \\ \cos\pi & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

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eig(A)

$Av = Sv$ → eigenvalue of A

$(SI - A)v = 0$

S: complex number

v: vector with elements $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

solve: $\det(SI - A) = 0$

$\Rightarrow$

$$Sv = S \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = S \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Interesting points are where $(SI - A)$ is <u>not</u> invertible.