Linear Systems 09/18/12

Last time

State space models, linearization

$$\dot{x} = f(x,u)$$
$$y = g(x,u)$$

$\xrightarrow[\text{around } (\bar{x},\bar{u})]{\text{linearization}}$

trajectory or equilibrium pt.

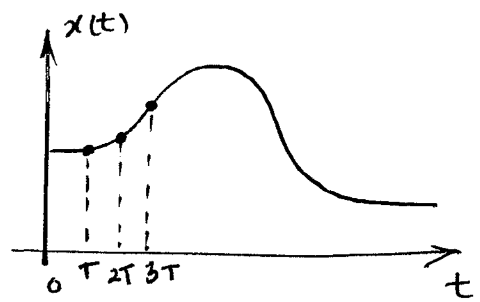
$$\dot{x} = Ax + Bu \quad (1)$$
$$y = Cx + Du \quad (2)$$

$$A = \left.\frac{\partial f}{\partial x}\right|_{(\bar{x},\bar{u})}$$

Today:

Discrete time system:

Q1: How can we discretize a cts time system?



T: sampling period

we'll pay attention to

$$\{x(0);\, x(T);\, x(2T);\, \dots\} = \{x(KT)\}$$

we'll use the flw notation

$$x(k) = x(KT)$$
$$x_k = x(KT)$$

Recall: $\dot{x}(t) := \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t}$

Forward Euler approx. $\dot{x}(KT) \approx \frac{x(KT+T) - x(KT)}{T} + O(T)$ (3)

(3) → (1) $\Rightarrow$ $\frac{x(K+1) - x(K)}{T} = A x(K) + B u(K)$

$$y(K) = C x(K) + D u(K)$$

$$x(K+1) = (I + T \cdot A) \cdot x(K) + T \cdot B \cdot u(K)$$
$$y(K) = C x(K) + D u(K)$$

$$x(K+1) = A_d \cdot x(K) + B_d u(K)$$
$$y(K) = C_d x(K) + D_d u(K)$$

In our case:
$$A_d = I + T \cdot A$$
$$B_d = T \cdot B$$
$$C_d = C$$
$$D_d = D$$

We will study systems of the form:

$$x(K+1) = A(K) x(K) + B(K) u(K)$$
$$y(K) = C(K) x(K) + D(K) u(K)$$

and we'll examine solutions of these systems.

Given an initial condition

$$x(0) = x_0$$

and input sequence,

$$u(0),\ u(1),\ \ldots,\ u(N)$$

we'll find $x(N)$ and $y(N)$.

We'll first study unforced systems (no input)

$$x(K+1) = A(K)\cdot x(K)$$
$$y(K) = C(K)x(K)$$

$$x(0) = x_0 \quad \text{given}$$

$$K=0 \xrightarrow{(1)} x(0+1) = A(0)\cdot x(0)$$

$$x(1) = A(0)\cdot x_0$$

$$K=1 \xrightarrow{(1)} x(1+1) = A(1)\,x(1)$$

$$x(2) = A(1)A(0)\,x_0$$

$$\vdots$$

$$x(K) = A(K-1)A(K-2)\cdots A(1)A(0)\cdot x_0$$

If initial time is $l \neq 0$

$$x(K) = \underbrace{A(K-1)\,A(K-2)\cdots A(l+1)\,A(l)}_{\Phi(k,l)}.x(l)$$

$\Phi(k, l) \Rightarrow$ state transition matrix.

k: final time

l: initial time

$$x(l) \xrightarrow{\Phi(K,l)} x(K)$$

$$\Phi(K,l) = A(K-1)\cdots A(l)$$

in particular, $l=0$: $\Phi(K,0) = A(K-1)\cdots A(0)$

In the time-invariant case: $A(K) = A = \text{const.}$ for all K

$$\Phi(K,0) = \mathbf{A}^K = \underbrace{A\cdot A\cdot\cdots\cdot A}_{K \text{ times}}$$

(not element-wise)

<u>Properties of $\Phi(K,l)$</u>:

1. $\Phi(l,l) = I$ (identity matrix) e.g. $I_{3\times3} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$ for a system of 3 states

2.
$$\begin{tikzcd} l \arrow[r, "\Phi(m,l)"'] \arrow[rr, bend left, "\Phi(K,l)"] & m \arrow[r, "\Phi(K,m)"'] & k \end{tikzcd}$$

$$\Phi(K,l) = \Phi(K,m).\Phi(m,l)$$

3. $$\Phi(K+1,l) = A(K)\cdot \underbrace{A(K-1)\cdots A(l)}_{\Phi(K,l)}$$

$$\begin{cases} \Phi(K+1,l) = A(K)\Phi(K,l) \\ \Phi(l,l) = I \end{cases}$$

Check these to see if a certain matrix can be a state transition matrix or not!

In cts time:

$$\begin{cases} \dfrac{\partial \Phi(t,\tau)}{\partial t} = A(t)\cdot\Phi(t,\tau) \\ \Phi(\tau,\tau) = I \end{cases}$$

Forced Systems (Systems with inputs)

$$x(K+1) = A(K)x(K) + B(K)u(K)$$

$x(K) =$ natural response + forced response

(arising from x_0) (arising from input (u))

$$x(1) = A(0)\,x(0) + B(0)\,u(0)$$

$$x(2) = A(1)\,x(1) + B(1)\,u(1) = A(1)\left[A(0)\,x(0) + B(0)\,u(0)\right] + B(1)\,u(1) =$$

$$= A(1)A(0)\,x(0) + A(1)B(0)\,u(0) + B(1)\,u(1)$$

$$x(3) = A(2)\,x(2) + B(2)\,u(2)$$

$$= A(2)A(1)A(0)\,x(0) + A(2)A(1)B(0)\,u(0) + A(2)B(1)\,u(1) + B(2)\,u(2)$$

which we can write as:

$$x(3) = \Phi(3,0)\,x(0) + \left[A(2)A(1)B(0) \;\vdots\; A(2)B(1) \;\vdots\; B(2)\right] \begin{bmatrix} u(0) \\ u(1) \\ u(2) \end{bmatrix}$$

Abstractly:

$$x(k) = \underbrace{\Phi(k,l)\,x(l)}_{\text{natural response}} + \underbrace{\sum_{m=l}^{k-1} \Phi(k,m+1)\cdot B(m)\,u(m)}_{\text{forced response}}$$

Time invariant case

$$x(k) = A^{(k-l)}\cdot x(l) + \sum_{m=1}^{k-1} \underbrace{A^{(k-m-1)}}_{\phi(k,m) = \phi(k-m)} \cdot B\cdot u(m)$$

$\mathcal{Z}$- transform

[Transform technique (help line)]
for DT, LTI systems

$\xrightarrow{\mathcal{Z}} F(z)$

$$\{f_0, f_1, \dots\} \xrightarrow[z \in \mathbb{C}]{\mathcal{Z}} F(z)$$

$$F(z) = \sum_{k=0}^{\infty} f_k \cdot z^{-k} = \sum_{k=0}^{\infty} f_k z^{-k} \qquad z \in \mathbb{C} \to \text{complex number}$$

or

$$\mathcal{Z}(f) = F(z) = \sum_{k=0}^{\infty} f_k z^{-k}$$

Properties of $\mathcal{Z}$-Transform:

1) linearity

$$\mathcal{Z}\{\alpha f + \beta g\} = \sum_{k=0}^{\infty} (\alpha \cdot f_k + \beta g_k) z^{-k} = \alpha \sum_{k=0}^{\infty} f_k z^{-k} + \beta \sum_{k=0}^{\infty} g_k z^{-k}$$

α, β scalars
f, g sequences

$$= \alpha \mathcal{Z}\{f\} + \beta \mathcal{Z}\{g\}$$

$$= \alpha F(z) + \beta G(z) \quad \checkmark$$