Lecture 6 09/25/12

Linear Systems

Last time:

Transform techniques for DT LTI systems [Z-transform]

Transfer function $H(z)$

Impulse & frequency responses

Today:

Solution to CT LTI systems

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \qquad (1)$$
$$y(t) = C(t)x(t) + D(t)u(t) \qquad (2)$$

Recall: $x(k+1) = A(k)\,x(k) + B(k)\,u(k)$

Solution was given by:

$$x(k) = \Phi(k,l)\cdot x(l) + \sum_{m=l}^{k-l} \Phi(k,m+1)\,B(m)\,u(m)$$

state transition matrix

initial condition

$$\Phi(k+1,l) = A(k)\,\Phi(k,l)$$

$$\Phi(l,l) = I$$

In CT we will propose solution to (1)

$$x(t) = \Phi(t,t_0)\,x(t_0) + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)\,d\tau \qquad (*)$$

We need to examine conditions on $\Phi(t,\tau)$ such that $(*)$ satisfies (I)

start with unforced state equation :

$$\dot{x}(t) = A(t)\,x(t)$$

* under what conditions (on $\Phi(t,t_0)$)

$$x(t) = \Phi(t,t_0)\,x(t_0) \qquad \boxed{*}$$

satisfies (I) ?

We need to check 2 things :

a) $x(t)$ given by $\boxed{*}$ satisfies initial conditions (i.c.s)

b) $x(t)$ " " " satisfies $\dot{x}(t) = A(t)x(t)$

a) from $\boxed{*}\Big|_{t=t_0}$ we have

$$x(t_0) = \Phi(t_0,t_0)\,x(t_0)$$

$$[\Phi(t_0,t_0) - I]\,x(t_0) = 0 \qquad \text{has to hold for all } x(t_0)$$

$$\Rightarrow \quad \boxed{\Phi(t_0,t_0) = I}$$

b) $\dfrac{d\boxed{*}}{dt} \Rightarrow \quad \dfrac{\partial \Phi(t,t_0)}{\partial t} \cdot x(t_0) = A(t)\,\Phi(t,t_0)\,x(t_0)$

this has to hold for all $x(t_0)$.

Thus from (a) & (b) we have :

$$\boxed{\begin{array}{ll} \text{(STM 1)} & \dfrac{\partial \Phi(t,t_0)}{\partial t} = A(t)\,\Phi(t,t_0) \\[2ex] \text{(STM 2)} & \Phi(t_0,t_0) = I \end{array}} \qquad \text{(STM)}$$

↳ A matrix valued function of two arguments (t,t_0) is called a state transition matrix if it solves system of equations (STM).

Note! For system with n states $(x(t) \in \mathbb{R}^n)$, we have $\Phi(t,t_0) \in \mathbb{R}^{n\times n}$ which means that there are n^2 unknowns in (STM).

Ex $\quad \dddot{y} + a_0(t) \cdot y(t) = u(t)$

count dots $\longrightarrow$ 2 states

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_0(t) & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Aside! $\quad$ If $a_0(t) = K_0 + \alpha \cos(\omega t)$: Mathieu equation

No explicit expression for $\Phi(t,t_0)$.

$$\Phi(t,t_0) = \left[\begin{bmatrix} \Phi_{11}(t,t_0) \\ \Phi_{21}(t,t_0) \end{bmatrix} \;\vdots\; \begin{bmatrix} \Phi_{12}(t,t_0) \\ \Phi_{22}(t,t_0) \end{bmatrix} \right]$$

partitioning columns

$$\begin{bmatrix} \dot{\Phi}_{11} & \dot{\Phi}_{12} \\ \dot{\Phi}_{21} & \dot{\Phi}_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_0(t) & 0 \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix}$$

$\longrightarrow$ 4 eq'ns 4 unknowns $\Phi_{ij}(t,t_0)$

$$\begin{bmatrix} \Phi_{11}(t_0,t_0) & \Phi_{12}(t_0,t_0) \\ \Phi_{21}(t_0,t_0) & \Phi_{22}(t_0,t_0) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We'll talk about numerical solution of (STM).

Forced case :

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

$$x(t_0) = x_0$$

$$x(t) = \Phi(t,t_0)\, x(t_0) + \int_{t}^{t} \Phi(t,\tau)\, B(\tau)\, u(\tau)\, d\tau$$

a) i.c.

$$x(t_0) = \Phi(t_0,t_0)\, x(t_0) + \cancelto{0}{\int_{t_0}^{t_0} \Phi(t_0,\tau)\, B(\tau)\, u(\tau)\, d\tau}$$

$\Rightarrow$ i.c. satisfies $\left(\Phi(t_0,t_0) = I \right)$

$$\frac{d \circledast}{dt} \Rightarrow \boxed{\dot{x}(t)} = \frac{\partial \Phi(t,t_0)}{\partial t} \cdot x(t_0) + \Phi(t,\tau)\, B(\tau)\, u(\tau) \Big|_{\tau = t}$$

$$+ \int_{t_0}^{t} \frac{\partial \Phi(t,\tau)}{\partial t}\, B(\tau)\, u(\tau)\, d\tau$$

$$= \overset{(STM1)}{A(t) \cdot \Phi(t,t_0)}\, x(t_0) + \underset{(STM2)}{\cancelto{I}{\Phi(t,t)}} \cdot B(t)\, u(t) +$$

$$+ \int_{t_0}^{t} \underset{(STM1)}{A(t)\, \Phi(t,\tau)}\, B(\tau)\, u(\tau)\, d\tau = \dots$$

$$\dots = A(t)\Big[\underbrace{\Phi(t,t_0)x(t_0) + \int_{t_0}^{t}\Phi(t,\tau)B(\tau)u(\tau)d\tau)}_{\boxed{*} \Rightarrow x(t)}\Big]$$

$$+ B(t)u(t)$$

$$= \boxed{A(t)x(t) + B(t)u(t)} \quad \text{q.e.d}$$

Time invariant problems 8

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$

A, B constant matrices

Ex $\quad \dot{x}(t) = a \cdot x(t) \quad ; \; a \in \mathbb{R}$

$$\frac{dx(t)}{dt} = a \cdot x(t) \Rightarrow \frac{dx(t)}{x(t)} = a \cdot dt$$

integrate : $\dots \longrightarrow x(t) = e^{a(t-t_0)}x(t_0)$

Propose : $\dot{x}(t) = Ax(t)$, $x(t) = e^{A(t-t_0)} \cdot x(t_0)$

Note! For time invariant systems state transition matrix only depends on a difference between t and t_0.

$$\Phi(t,t_0) = \Phi(t-t_0)$$

$$\begin{cases} \dfrac{\partial \Phi(t,t_0)}{\partial t} = A\,\Phi(t,t_0) \\ \Phi(0) = I \end{cases}$$

Question? $\rightarrow$ Can we extend e^{at} from scalars to matrices? If so what is a proper generalization?

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\Rightarrow \quad x_1(t) = e^{-t} x_1(0)$$

$$x_2(t) = e^{-2t} x_2(0)$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}}_{\text{Satisfies (STM)}} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

"e^{At}" : not determined by exponentials of individual elements of matrix $A.t = \begin{bmatrix} -t & 0 \\ 0 & -2t \end{bmatrix}$

$\Phi(t,0) = \Phi(t-\mathbf{0})$

because if it was we would have : $\begin{bmatrix} e^{-t} & 1 \\ 1 & e^{-2t} \end{bmatrix}$

Recall: definition of e^{at},

$$e^{at} = \sum_{k=0}^{\infty} \frac{(at)^k}{k!} = 1 + \frac{at}{1} + \frac{(at)^2}{2!} + \cdots$$

Proper definition of matrix exponential :

$$e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!} = I + At + \frac{A^2 t^2}{2!} + \cdots$$

(STM2) $$e^{A.0} = I + \frac{A.0}{1} + \frac{A^2.0^2}{2!} + \cdots = I$$

(STM1) $$\frac{d e^{At}}{dt} = A e^{At}$$ (check this!)