Lecture 7 09/27/12

Linear Systems

- e^{At}
- Transfer functions

$$e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!} = I + At + \frac{(At)^2}{2!} + \cdots$$

Note! For LTI systems

$$\Phi(t,t_0) = \Phi(t-t_0) = e^{A(t-t_0)}$$

1. $$\boxed{\Phi(t_0,t_0) \stackrel{?}{=} e^{A(t_0-t_0)} = e^{A\cdot 0} = I + A\cdot 0 + \frac{(A\cdot 0)^2}{2} + \cdots \stackrel{?}{=} I}$$

2. $$\frac{\partial e^{A(t-t_0)}}{\partial t} = \partial/\partial t \left\{ I + A(t-t_0) + \frac{1}{2} A^2 (t-t_0)^2 + \cdots \right\}$$
$$= 0 + A\cdot 1 + \frac{1}{\cancel{2}} \cdot A^2 \cdot \cancel{2}(t-t_0) + \cdots$$
$$= A\left[I + A(t-t_0) + \cdots\right] = A e^{A(t-t_0)}$$

(1) & (2) $\Rightarrow$ $e^{A(t-t_0)}$ is indeed <u>the</u> state transition matrix of $\dot{x}(t) = A \cdot x(t)$

Note! Without loss of generality, in LTI systems we can set $t_0 = 0$: $\Phi(t,0) = e^{At}$

the choice of initial time is not important, the difference between two time instances is important.

Now we must specialize the equation

$$x(t) = \Phi(t,t_0)\,x(t_0) + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)\,d\tau \quad (*)$$

for LTI systems:

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)} \underbrace{B(\tau)}_{B \text{ for LTI}} u(\tau)\,d\tau$$

for $t_0 = 0$

$$x(t) = e^{At}x(0) + \int_{0}^{t} e^{A(t-\tau)} B \cdot u(\tau)\,d\tau$$

Output: $y(t) = C\,x(t) + D \cdot u(t)$

$$y(t) = Ce^{At} \cdot x(0) + \int_{0}^{t} C \cdot e^{A(t-\tau)} \cdot Bu(\tau)\,d\tau + Du(\tau) \quad (I)$$

$$y(t) = C \cdot e^{At} x(0) + \int_0^t \left[C e^{A(t-\tau)} \cdot B + D\delta(t-\tau) \right] u(\tau)\, d\tau$$

conceptually:

$$y(t) = C \cdot e^{At} x(0) + \int_0^t H(t-\tau) u(\tau)\, d\tau$$

Ex

$$H(s) = \frac{s+2}{s+1} = \frac{s+1+1}{s+1} = \frac{1}{s+1} + 1$$

this will bring D term into s.s. equations

Properties of $\Phi(t,t_0)$

1. Solves $\dfrac{\partial \Phi(t,t_0)}{\partial t} = A(t)\Phi(t,t_0)$

   and $\Phi(t_0,t_0) = I$

2. $\Phi(t,t_0) = \Phi(t,\tau)\,\Phi(\tau,t_0)$

$$\begin{tikzcd} t_0 \arrow[r, "\Phi(\tau,t_0)"] \arrow[rr, bend right, "\Phi(t,t_0)"'] & \tau \arrow[r, "\Phi(t,\tau)"] & t \end{tikzcd}$$

3. $\Phi(t,t_0)$ is invertible

$$0 \neq \det(\Phi(t,t_0)) = e^{\int_{t_0}^{t} \text{trace}(A(\tau))d\tau}$$

$$\text{trace}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 1+4 = 5$$

4. $e^{A(t_1+t_2)} = e^{At_1} \cdot e^{At_2}$

$e^{(A_1+A_2)t} \overset{?}{=} e^{A_1 t} e^{A_2 t}$ $\longrightarrow$ only holds iff $A_1A_2 = A_2A_1$ (if & only if)
or A_1 and A_2 commute

Matlab: expm (A)

Note! In general we need to compute $\Phi(t,t_0)$ numerically.

* In your HW, $A(t) = \begin{bmatrix} 0 & t \\ 0 & 2 \end{bmatrix}$ you could find an explicit expression for $\Phi(t,t_0)$.

Note! Partition $\Phi(t,t_0)$ in terms of its columns

$$x(t) = \Phi(t,t_0)x(t_0) =$$

$$= \left[\phi_1(t,t_0) \;\middle|\; \phi_2(t,t_0) \;\middle|\; \cdots \;\middle|\; \phi_n(t,t_0) \right] \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \\ \vdots \\ x_n(t_0) \end{bmatrix}$$

$$\begin{bmatrix} x(t) \end{bmatrix}_{n\times 1} = \sum_{i=1}^{n} \begin{bmatrix} \Phi_i(t,t_0) \end{bmatrix} \cdot x_i(t_0)$$

Note! $x(t)$ is determined by linear combinations of matrix $\Phi(t,t_0)$. Corresponding weights are determined by, $x_i(t_0)$, $i=1,\dots,n$.

In particular

$$x_m(t_0) = 1 \quad ; \quad x_j(t_0) = 0$$

$m \in \{1,\dots,n\}$ $\qquad j \neq m$

$$x(t) = \Phi_m(t,t_0)$$

Thus, $\Phi_m(t,t_0)$ can be determined by solving the following differential eq'n :

$$\dot{\Phi}_m(t,t_0) = A(t) \cdot \Phi_m(t,t_0)$$

$$\Phi_m(t_0,t_0) = e_m \rightsquigarrow e_m = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \rightarrow m\text{th position}$$

$m = \{1,\dots,n\}$

mth unit vector in $\mathbb{R}^n$

Thus we need to run n- simulations with initial conditions

$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$$

Laplace Transform :

$$f(t) \xrightarrow[s \in \mathbb{C}]{\mathcal{L}} F(s)$$

Recall : Z-transform was of the form,

$$F(z) = \sum_{k=0}^{\infty} f(k) \cdot z^{-k}$$

we now have :

$$F(s) = \int_0^{\infty} f(t)\, e^{-st}\, dt$$

1) Linearity

2) $\mathcal{L}\left\{\frac{df}{dt}\right\} = sF(s) - f(0)$

initial condition

If all i.c.'s are $\equiv 0$

$$\mathcal{L}\left\{\frac{d^m f}{dt^m}\right\} = s^m F(s)$$

3. $\mathcal{L}\left\{ \int_0^t h(t-\tau)u(\tau)d\tau \right\} = H(s) \cdot U(s)$

4. $\mathcal{L}\{\delta(t)\} = 1$

delta function: $\int f(t-\tau)\delta(\tau)d\tau = f(t)$

$$\dot{x}(t) = Ax(t) + Bu(t) \quad \cdots \quad (1)$$
$$y(t) = Cx(t) + Du(t) \quad \cdots \quad (2)$$

$\mathcal{L}(1) \Rightarrow$ $\quad sX(s) - x(0) = AX(s) + BU(s)$

$$X(s) = \underbrace{(sI-A)^{-1}}_{\text{resolvent}} x(0) + (sI-A)^{-1}BU(s) \quad (3)$$

$(3) \rightarrow \mathcal{L}(2)$:

$$Y(s) = C(sI-A)^{-1} \cdot x(0) + \underbrace{[C(sI-A)^{-1}B + D]}_{\text{transfer function}} U(s) \quad (\text{II})$$

from (I) & (II) we can conclude that

$$e^{At} = \mathcal{L}^{-1}\{(sI-A)^{-1}\}$$

Impulse response: $H(t) = C \cdot e^{At}B + D\delta(t) \Rightarrow H(t) = \mathcal{L}^{-1}\{H(s)\}$