Lecture 8 10/04/12

Linear Systems

- Solutions to CT LTI systems
- Matrix exp.
- Laplace Transform
- Transfer function

Solution to :

$$\dot{x} = Ax + Bu$$

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)} B\, u(\tau)\, d\tau$$

$$y = Cx + Du$$

$$y(t) = Ce^{At}x_0 + \int_0^t \underbrace{[Ce^{A(t-\tau)}B + D\,\delta(t-\tau)]}_{H(t-\tau)} u(\tau)\, d\tau$$

We used Laplace transform to introduce notion of a transfer function

[A mapping from inputs to the outputs in a complex domain]

$$Y(s) = C\underbrace{(sI-A)^{-1}}_{\text{resolvent}} x_0 + H(s)\,U(s)$$

where :

$$H(s) = \mathcal{L}\{H(t)\} = C\underbrace{(sI-A)^{-1}}_{R}B + D$$

$$Y(s) = H(s)\,U(s)\Big|_{x_0 = 0}$$

$\Rightarrow$ Transfer function is the laplace transform of the impulse response.

Now if $x_0 = 0 \Rightarrow y(t) = \int_0^t C e^{A(t-\tau)} B u(\tau)\, d\tau + D u(t)$

if $u(t) = \delta(t) \Rightarrow$

$$y(t) = C e^{At} B + D\delta(t) = H(t) \quad \text{impulse response}$$

Note! If we know the impulse response then the response to an arbitrary u is given by

$$y(t) = \int_0^t H(t-\tau)\, u(\tau)\, d\tau \Big|_{x_0 \equiv 0}$$

Also Note! since $\mathcal{L}\{\delta(t)\} = 1$

$$\Rightarrow \quad Y(s) = H(s) \Big|_{U(s)=1}$$

Ex

Double integrator

$$\ddot{y}(t) = u(t)$$

acceleration

force input

Unforced system :

$$\ddot{y}(t) = 0 \Rightarrow \dot{y}(t) = c_1 \Rightarrow y(t) = c_1 t + c_2$$

$$\dot{y}(0) = c_1 \qquad y(0) = c_1 \cdot 0 + c_2$$

$$\Rightarrow y(0) = c_2$$

$$y(t) = \dot{y}(0)t + y(0)$$

A state-space model :

$$\left.\begin{aligned} x_1 &= y \\ x_2 &= \dot{y} \end{aligned}\right\} \Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

In the unforced cases :

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x(t) = e^{At} x(0)$$

$$e^{At} = I + At + \frac{A^2 t^2}{2!} + \cdots$$

$$A \cdot A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$e^{At} = I + At = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

Altenatively :

$$e^{At} = \mathcal{L}^{-1} \{ (sI - A)^{-1} \}$$

$$(sI - A)^{-1} = \frac{1}{\det(sI - A)} \operatorname{adj}(sI - A)$$

$$sI - A = s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}$$

$$\det(sI-A) = s^2 - 0\cdot 1 = s^2$$

$$\operatorname{adj}(sI-A) = \begin{pmatrix} s & 1 \\ 0 & s \end{pmatrix}$$

$$\Rightarrow (sI-A)^{-1} = \frac{1}{s^2}\begin{pmatrix} s & 1 \\ 0 & s \end{pmatrix} = \begin{pmatrix} 1/s & \frac{1}{s^2} \\ 0 & 1/s \end{pmatrix}$$

$$\Rightarrow e^{At} = \mathcal{L}^{-1}\{(sI-A)^{-1}\} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

position

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_1(0) + t x_2(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}$$

velocity

$$\ddot{y}(t) = u(t) \xrightarrow{\mathcal{L}} s^2 Y(s) - s y(0) - \dot{y}(0) = U(s)$$

0 if $u(t)$ is not there

$$Y(s) = \frac{1}{s} y(0) + \frac{1}{s^2}\dot{y}(0) + \frac{1}{s^2}U(s)$$

0

$$y(t) = y(0) + t\dot{y}(0) + 0$$

but if i.c. $= 0$

$$s^2 Y(s) - s y(0) - \dot{y}(0) = U(s)$$

$$\text{i.c.} = 0 \longrightarrow s^2 Y(s) = U(s) \Rightarrow Y(s) = \frac{1}{s^2}U(s) \Rightarrow H(s) = \frac{1}{s^2}$$

Yet another way of obtaining the transfer function:

$$\boxed{H(s)} \stackrel{!}{=} C(sI-A)^{-1}B+D = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}1/s & 1/s^2 \\ 0 & \frac{1}{s}\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} + 0$$

$$= \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{s}0+\frac{1}{s^2}\cdot 1 \\ 0.0+\frac{1}{s}\cdot 1\end{bmatrix} = \boxed{\frac{1}{s^2}}$$

$$H(s) = 1/s^2 \xrightarrow{\mathcal{L}^{-1}} H(t) = t\,\mathbb{1}(t) \quad \text{impulse response}$$

→ We'll start talking about properties of solutions (e.g. equilibrium points) soon.

Recall: for $\dot{x} = f(x)$

$\bar{x}$ is an equilibrium pt. if it satisfies $f(\bar{x}) = 0$

for $\dot{x} = Ax \Rightarrow A\cdot\bar{x} = 0$

In othe words, eq. points of CT LTI systems are characterized by a nullspace of matrix A.

$Null(A) = \{\bar{x} \text{ st } A\bar{x} = 0\} \Rightarrow$

$Null(A) = \{0\}$

More interesting things happen if $\det A = 0$

(non-trivial null space) $\longrightarrow$ ∞ many eq. points

In double integrator example :

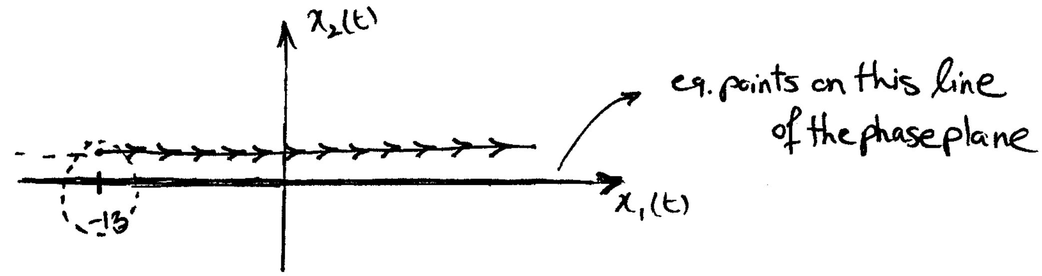
$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \det(A) = 0 \Rightarrow \text{non trivial null space}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} 0\bar{x}_1 + \bar{x}_2 = 0 \\ 0 \cdot \bar{x}_1 + 0\bar{x}_2 = 0 \end{array} \Rightarrow \begin{bmatrix} \bar{x}_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $\bar{x}_2 = 0$
$\bar{x}_1$: arbitrary real number

$$\bar{x} = Null(A) = \left\{ \begin{bmatrix} \bar{x}_1 \\ 0 \end{bmatrix}, \ \bar{x}_1 \in \mathbb{R} \right\}$$

In phase plane :



If you start from any point on the blue line with no forcing you are going to stay there.

we had :

$$\begin{cases} x_1(t) = x_1(0) + t\,x_2(0) \\ x_2(t) = x_2(0) \end{cases}$$

but if you start from any point in the neighborhood of the equilibrium point $(x_1(t), x_2(t)) = (-13, 0)$ we are not going to stay there $\longrightarrow$ this equilibrium point is not stable!

$\longrightarrow$ we will formalize this soon.

Recall : $\dot{x}(t) = a \cdot x(t)$; $x(t) \in \mathbb{R}$

recall

$$x(t) = e^{at} \cdot x(0)$$

For vector valued problems:

$$\dot{x}(t) = A\,x(t) \quad ; \; x(t) \in \mathbb{R}^n \qquad (*)$$

Under what condition on complex number s and vector v would

$$x(t) = e^{st} \cdot v \quad , \qquad \text{solve } (*)$$

$$x(0) = e^{s0} \cdot v = v$$

$$(*) \Rightarrow \frac{d}{dt}(e^{st} v) = A \cdot e^{st} \cdot v$$

$$s e^{st} \cdot v = A \cdot e^{st} v$$

$$e^{st}(sI - A)v = 0$$

$$(sI - A)v = 0$$