Last time: double integrator

10/5/12

$$\ddot{y}(t) = u(t)$$

solution to (1) given by:

$$y(t) = y_h(t) + y_p(t)$$

y_h comes from I.C.s; y_p comes from inputs

$y_H(t)$ = solution to unforced system

i.e. $\ddot{y}(t) = 0 \xrightarrow{\mathcal{L}} \underbrace{s^2}\cdot Y(s) = 0$

$f(s) = s^2$: characteristic polynomial

$= \det(sI - A)$

$y_H(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t}$ $\Big\{$ solutions to $s^2 = 0$: $s_1 = 0$, $s_2 = 0$

$\Downarrow$

correct if $s_1 \neq s_2$

* for $s_1 = s_2 = s$

$$y_H(t) = c_1 e^{st} + c_2 t e^{st}$$
$$= c_1 e^{0 \cdot t} + c_2 t e^{0 \cdot t}$$

$$\boxed{y_H(t) = c_1 + c_2 t}$$

Eigenvalue Decomposition of a matrix $A \in \mathbb{R}^{n \times n}$

$$Av = \lambda v \qquad \begin{array}{l} \lambda \in \mathbb{C} \\ v \in \mathbb{C}^n \end{array}$$

$$\begin{bmatrix} A \end{bmatrix}\begin{bmatrix} v \end{bmatrix} = \lambda \begin{bmatrix} v \end{bmatrix}$$

want to find solutions to this eqn.

* Any pair (λ, v), where $\lambda \in \mathbb{C}$ (in general) and $v \in \mathbb{C}^n$, that solves above equation with $v \neq 0$ is called eigenvalue, eigenvector pair

In other words, we will be looking for complex numbers λ for which there is a nontrivial solution $v \neq 0$ to $Av = \lambda v$

$$Av = \lambda v \iff \lambda v - Av = 0$$

$$(\lambda I - A)v = 0$$ want to find nontrivial solutions

If $\lambda \in \mathbb{C}$ such that $(\lambda I - A)$ is invertible, then $v = 0$ is the only solution to $Av = \lambda v$

* We are interested only in $\lambda \in \mathbb{C}$ s.t. $(\lambda I - A)$ is not invertible

Thus: eigenvalues of A are determined from solutions to

$$\underline{\det(\lambda I - A) = 0}$$

$\boxed{\text{EX}}$

$$A = \begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix}$$

$$\lambda I - A = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 5 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda & -1 \\ 5 & \lambda+6 \end{bmatrix}$$

$$\det(\lambda I - A) = \lambda(\lambda+6) + 5$$

$$0 = \lambda^2 + 6\lambda + 5$$

* obtain eigenvalues

continued

$$= (\lambda+5)(\lambda+1)$$

$$\lambda = -1, -5$$

eigen vectors associtated w/

$$\left.\begin{array}{l} \lambda_1 = -1 \\ \lambda_2 = -5 \end{array}\right\} \begin{array}{l} \text{solve} \\ (\lambda_i I - A) v_i = 0 \end{array}$$

for $\lambda_1 = -1$

$$\left(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix} \right) \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

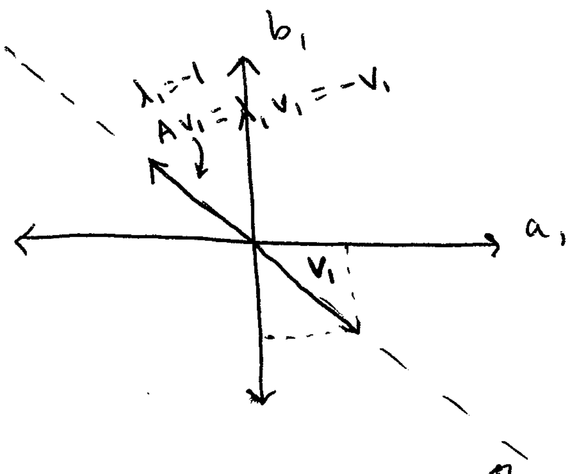
$$-a_1 - b_1 = 0 \quad \ldots (1)$$

$$5a_1 + 5b_1 = 0 \quad \ldots (2)$$

$$(1) \quad b_1 = -a_1 \quad \ldots (3)$$

$$(3) \Rightarrow (2) \quad 5a_1 - 5a_1 = 0 \quad \checkmark$$

$$\therefore \; v_1 = \begin{bmatrix} a_1 \\ -a_1 \end{bmatrix} \qquad a_1 \in \mathbb{R} \text{ or } a_1 \in \mathbb{C}$$



any ~~eigen~~ vector along this line
is a legitimate eigenvector

For $A \in \mathbb{R}^{n \times n}$

$$A \cdot v_i = \lambda_i v_i \; ; \; i = 1, \ldots, n$$

$$A \cdot [v_1 \mid v_2 \mid \cdots \mid v_n] =$$

Let's rewrite this as:
(keep dimensions in mind)

~~eigenvalues on the main diagonal~~

eigen values on main diagonal

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} v_1 & v_2 & v_3 & \cdots & v_n \end{bmatrix} \cancel{\begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}} = \underbrace{\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}}_{\Lambda}$$

From example,

$$\begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix} \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -5 \end{bmatrix}$$

If matrix A has a linearly independent set of eigenvectors $\{v_1, \ldots, v_n\} \Rightarrow V = [v_1 \mid \cdots \mid v_n]$ is invertible

$$A \cdot V = V \cdot \Lambda$$

$$A \cdot V \cdot V^{-1} = V \cdot \Lambda V^{-1}$$

$$\boxed{A = V \Lambda V^{-1}}$$

diagonal coordinate form of a matrix

rearranging, $\Lambda = V^{-1} A V$

* Note: Not every matrix is diagonalizable

$\boxed{\text{EX}}$ $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ eigenvalues are 0, 0
- not independent set of eigenvectors

* Symmetric matrices can always be diagonalized

$$A = A^T$$

$\boxed{EX}$ $A = \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}$

* Normal matrices can always be diagonalized

$$A \cdot A^T = A^T A$$

* take a look at your textbook for Jordan diagonalizable matrix

<u>Note</u>: Any matrix can be brought into block diagonal form

$$\begin{bmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_n \end{bmatrix}$$ J_i : Jordan block

What does this have to do with linear systems?

$$\dot{x}(t) = Ax(t) + Bu(t) \quad \ldots (1)$$
$$y(t) = Cx(t) + Du(t) \quad \ldots (2)$$

Assume that A is diagonalizable

$\{V_1, \ldots, V_n\}$ are linearly independent

Introduce a new variable $z(t)$

$$z(t) = V^{-1}x(t) \iff x(t) = \cancel{A}V \cdot z(t) \quad \ldots (3)$$

$(3) \to (1) \implies V \cdot \dot{z}(t) = A \cdot V \cdot z(t) + Bu(t)$

$(3) \to (2) \implies y(t) = C \cdot V \cdot z(t) + Du(t)$

multiply by V^{-1}

$$\dot{z}(t) = \underbrace{V^{-1}AV}_{\bar{A}} z(t) + \underbrace{V^{-1}B}_{\bar{B}} \cdot u(t)$$

$$y(t) = \underbrace{C \cdot V}_{\bar{C}} \cdot z(t) + \underbrace{D}_{\bar{D}} \cdot u(t)$$

In new coordinates,

$$\dot{z}(t) = \bar{A}z(t) + \bar{B}u(t)$$
$$y(t) = \bar{C}z(t) + \bar{D}u(t)$$

where, $\bar{A} = V^{-1}AV$

$\bar{B} = V^{-1}B$

$\bar{C} = CV$

$\bar{D} = D$

Note: V can be any invertible matrix

$\boxed{\text{EX}}$ Mass-Spring System

$$x(t) = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}$$

Q: Is $z(t) = \begin{bmatrix} y(t) + \dot{y}(t) \\ y(t) - \dot{y}(t) \end{bmatrix}$ a legitimate choice of new coordinates?

yes, as long as matrix that relates them is invertible

$$z(t) = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}}_{x(t)} \quad \checkmark$$

$$\det V = -2$$

<u>Question</u>: Did transfer function from $u \to y$ change?

$$\begin{aligned}
\bar{H}(s) &= \bar{C} \cdot (sI - \bar{A})^{-1} \bar{B} + \bar{D} \\
&= C \cdot V (sI - V^{-1}AV)^{-1} V^{-1}B + D \\
&= C \cdot V [s \cdot V^{-1}V - V^{-1}AV]^{-1} V^{-1}B + D \\
&= C \cdot V \cdot [V^{-1}(sI - A) V]^{-1} V^{-1} B + D \\
&= C \cdot \underbrace{V \cdot V^{-1}}_{I} (sI - A)^{-1} \underbrace{V V^{-1}}_{I} B + D \\
&= CI (sI - A)^{-1} IB + D \\
&= C (sI - A)^{-1} B + D \quad \checkmark = H(s)
\end{aligned}$$

nothing has changed for transfer function!

input $\longrightarrow$ output properties are held