Lecture 10

10/09/12

Linear Systems

E-value decomposition

eigenvalues/eigenvectors

$$A v_i = \lambda_i v_i \qquad i = 1, \dots, n$$

- Diagonalization ; $A = V \Lambda V^{-1}$ where $V = [v_1 \; v_2 \cdots v_n]$

equivalent state-space representations

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Given $T \in \mathbb{R}^{n \times n}$; $\det T \neq 0$

$$z(t) = T^{-1} x(t) \iff x(t) = T z(t)$$

Ex $\quad x(t) = \begin{bmatrix} p(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} \text{position} \\ \text{velocity} \end{bmatrix}$

$$z(t) = \begin{bmatrix} p(t) + v(t) \\ p(t) - v(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{T^{-1}} \begin{bmatrix} p(t) \\ v(t) \end{bmatrix}$$

$$\dot{x} = T\dot{z} = ATz + Bu$$
$$y = CTz + Du$$

$$\dot{z} = T^{-1}ATz + T^{-1}Bu$$
$$y = CTz + Du$$

In new coordinates: $\quad \dot{z}(t) = \bar{A}z(t) + \bar{B}u(t)$

$$y(t) = \bar{C}z(t) + \bar{D}\,u(t)$$

where,
$$\bar{A} = T^{-1}AT$$
$$\bar{B} = T^{-1}B$$
$$\bar{C} = C.T$$
$$\bar{D} = D$$

Note! Last time we showed that transfer function does not change, i.e., it is invariant under coordinate transformation T.

Q.: How about e-values of A?

$$\left.\begin{array}{l} A v_i = \lambda_i v_i \\ A = T\bar{A}T^{-1} \end{array}\right\} \Rightarrow T\bar{A}T^{-1}v_i = \lambda_i v_i$$

$$\bar{A}\underbrace{(T^{-1}v_i)}_{\xi_i} = \lambda_i(T^{-1}v_i)$$

$$\bar{A}\xi_i = \lambda_i \xi_i$$

Conclusion : e-values didn't change

e-vectors did change :

$$\underbrace{\xi_i}_{\text{e-vectors of } \bar{A}} = T^{-1}\underbrace{v_i}_{\text{e-vectors of } A}$$

As shown last time :

$$A v_i = \lambda_i v_i$$

$$\begin{bmatrix} & A & \end{bmatrix}\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} = \overbrace{\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}}^{V}\overbrace{\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}}^{\Lambda}$$

if $\{v_1, v_2, \ldots, v_n\}$ are linearly independent, then $V = [v_1 \cdots v_n]$ is invertible and we can write

$$A = V \Lambda V^{-1}$$

Thus if we select $T = V \Rightarrow \bar{A} = V^{-1} A V = \Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$

Thus in new coordinates (z-coordinates) the unforced system is governed by,

$$\begin{bmatrix} \dot{z}_1(t) \\ \vdots \\ \dot{z}_n(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} z_1(t) \\ \vdots \\ z_n(t) \end{bmatrix}$$

$$\Longleftrightarrow \qquad \dot{z}_i(t) = \lambda_i z_i(t) \qquad i = 1, \dots, n$$

$$\boxed{z_i(t) = e^{\lambda_i t} z_i(0)}$$

$$\begin{bmatrix} z_1(t) \\ \vdots \\ z_n(t) \end{bmatrix} = \underbrace{\begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix}}_{e^{\Lambda t} = e^{\bar{A} t}} \begin{bmatrix} z_1(0) \\ \vdots \\ z_n(0) \end{bmatrix}$$

$$z(t) = e^{\Lambda t} \cdot z(0)$$

$$V^{-1} x(t) = e^{\Lambda t} V^{-1} x(0)$$

$$x(t) = \underbrace{V e^{\Lambda t} V^{-1}}_{e^{At}} x(0) \qquad \Rightarrow \qquad \boxed{e^{At} = V e^{\Lambda t} V^{-1}}$$

Matlab:

$$[V, \underset{\Lambda}{D}] = eig(A)$$

forexample for $t=54$

$$expA54 = V \cdot \underbrace{expm(D \cdot 54)}_{\begin{bmatrix} e^{\lambda_1 54} & & \\ & \ddots & \\ & & e^{\lambda_n 54} \end{bmatrix}} V^{-1}$$

Note! Not every matrix is diagonalizable:

Ex $A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$

e-values $\rightarrow \lambda$ with algebraic multiplicity 2 but e-vectors are not linearly independent! Therefore V is not invertible here.

E-vectors? $(\lambda I - A)v = 0$

$$\lambda_1 = \lambda_2 = \lambda$$

$$\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \right) \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\beta = 0$

$\alpha =$ arbitrary $\quad \Rightarrow \quad V_1 = V_2 = \begin{bmatrix} * \\ 0 \end{bmatrix}$

Fact! Any matrix can be brought into a block-diagonal form: $\begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_m \end{bmatrix}$ where matrices J_i are either Jordan blocks or diagonal matrices.

Ex $\begin{bmatrix} 3 & & \\ & 5 & \\ & & \begin{bmatrix} -4 & 1 & 0 \\ 0 & -4 & 1 \\ 0 & 0 & -4 \end{bmatrix} \end{bmatrix}$ $\quad J_1 = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$ $\quad J_2 = \begin{bmatrix} -4 & 1 & 0 \\ 0 & -4 & 1 \\ 0 & 0 & -4 \end{bmatrix}$

Note! If matrix A is normal, i.e., $AA^T = A^TA$

Then e-vectors of A are orthogonal to each other and matrix A can be diagonalized via unitary coordinate transformation.

$$\begin{cases} \text{Orthogonal matrix}: & A^{-1} = A^T \\ \text{Unitary matrix}: & A^{-1} = A^* \end{cases}$$

Note! $V \in \mathbb{R}^{n \times n}$ is orthogonal $\Leftrightarrow$ $V \cdot V^T = V^T \cdot V = I$

(i.e. $V^{-1} = V^T$)

Similarly, $V \in \mathbb{C}^{n \times n}$ is unitary $\Leftrightarrow$ $V . V^* = V^* . V = I$

V^* is the complex conjugate transpose of a matrix V

(in Matlab V^* is V')

<u>Ex</u>

$$V = \begin{bmatrix} 1+j & 3-2j \\ 5+6j & 3 \end{bmatrix} \Rightarrow \underset{(\bar{V})^T}{V^*} = \begin{bmatrix} 1-j & 5-6j \\ 3+2j & 3 \end{bmatrix}$$

For normal matrices :

$$A = V \Lambda V^* = \begin{bmatrix} V_1 & V_2 & \cdots & V_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} V_1^* \\ \vdots \\ V_n^* \end{bmatrix}$$

→ Action of matrix A on an arbitrary vector ψ in $\mathbb{R}^n$ ($\psi \in \mathbb{R}^n$)

$$A \cdot \psi = \begin{bmatrix} V_1 & \cdots & V_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} V_1^* \\ \vdots \\ V_n^* \end{bmatrix} \psi = \sum_{i=1}^{n} \underset{\text{scalar}}{\lambda_i} v_i \underbrace{V_i^* \psi}_{\text{scalar}} \quad \therefore$$

linear combination of v_i's which are orthogonal to each other

$Q: \ \psi = V_m$

$$V_i^* V_m = \begin{cases} 1 & i = m \\ 0 & i \neq m \end{cases}$$

$$A V_m = \lambda_m V_m$$

Modal representation of responses of unforced LTI systems:

$$\dot{x} = A x \quad \Rightarrow \quad x(t) = e^{At} x(0) = V e^{\Lambda t} V^* x(0)$$

$$= \sum_{i=1}^{n} e^{\lambda_i t} V_i \underbrace{V_i^* x(0)}_{\text{modal contribution of } x(0)}$$

in discrete time replace with λ_i^t

If matrix A is not normal but still diagonalizable then it can be written as:

$$A = V \Lambda W^* \quad \text{where} \quad W^* = \begin{bmatrix} w_1^* \\ \vdots \\ w_n^* \end{bmatrix}$$

w_i : left e-vectors of A

$$w_i^* A = \lambda_i w_i^*$$

$$(A^* w_i = \bar{\lambda}_i w_i)$$

$$\Rightarrow \quad x(t) = \sum_{i=1}^{n} e^{\lambda_i t} V_i \underbrace{w_i^* x(0)}_{\tau_i}$$

Difference: V_i's are no longer orthogonal to each other.

Note! w_i's can always be selected s.t. $w_i^* V_j = \begin{cases} 1, & i = j \\ 0 & i \neq j \end{cases}$