Lecture 11 10/11/12

Linear Systems

last time:

Modal representation of responses of (Unforced) LTI systems

Fact: If A is diagonalizable then the solution of

$$\dot{x}(t) = Ax(t) \quad , \quad x(0) = x_0$$

can be written as:

$$\begin{bmatrix} x(t) \end{bmatrix} = \sum_{i=1}^{n} e^{\lambda_i t} \begin{bmatrix} v_i \end{bmatrix} \begin{bmatrix} \quad w_i^* \quad \end{bmatrix} \begin{bmatrix} x_0 \end{bmatrix}$$

Note! $Av_i = \lambda v_i$; v_i : right e-vectors (e-vectors)

$w_i^* A = \lambda_i w_i^*$; w_i : left e-vectors

$(A^* w_i = \bar{\lambda}_i w_i)$

Fact: w_i's can always be selected st.

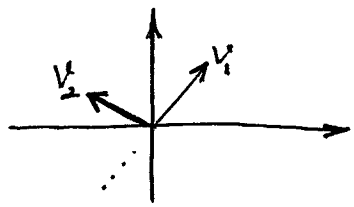
$$w_i^* v_j = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Q: $x_0 = v_m \Rightarrow \quad x(t) = e^{\lambda_m t} \cdot v_m$

If matrix A is normal $A^T A = A A^T$

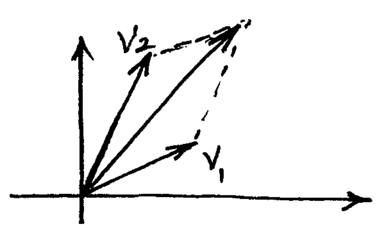
then v_i's are mutually orthogonal and $w_i = v_i$

e.g. for $n=2$



$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 e^{\lambda_1 t} \begin{bmatrix} v_1 \end{bmatrix} + \underbrace{c_2}_{w_2^* x_0} e^{\lambda_2 t} \begin{bmatrix} v_2 \end{bmatrix}$$

For non-normal matrices: we can have e-vectors that are far from being orthogonal (nearly parallel)



e.g. $A = \begin{bmatrix} -1 & 0 \\ k & -2 \end{bmatrix}$ $\quad k=0$ A diagonal

as k grows A becomes more ill conditioned and e-vectors are going to move from orthogonal to almost parralel.

For systems with inputs:

$$x(t) = e^{At} x_0 + \int_0^t e^{A(t-\tau)} \underbrace{B \cdot u(\tau)}_{q(\tau)} d\tau$$

$$= \sum_{i=1}^{n} e^{\lambda_i t} v_i w_i^* x_0 + \sum_{i=1}^{n} \int_0^t e^{\lambda_i (t-\tau)} v_i w_i^* q(\tau) d\tau$$

Recall: if matrices are diagonalizable we can have

$$\dot{x}(t) = A x(t)$$

can be brought into the following form: $\dot{z}(t) = \Lambda z(t)$

where $A = V \Lambda V^{-1}$

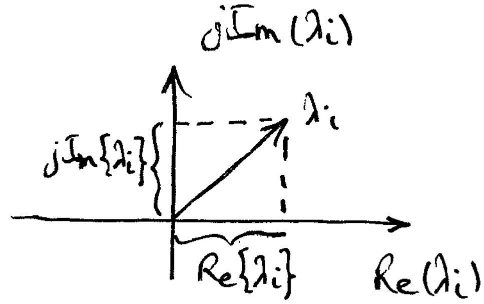
$$\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \Rightarrow z_i(t) = e^{\lambda_i t} z_i(0) \quad ; \quad t \in [0, \infty)$$

$$i = 1, \ldots, n$$

In D.T.

$$z_i(t) = \lambda_i^T z_i(0) \qquad ; \ t = 0, 1, 2, \ldots$$

$\lambda_i \in \mathbb{C}$



In C.T.

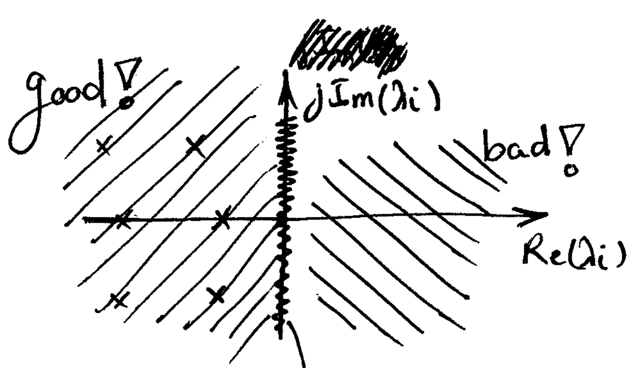
$$z_i(t) = e^{(Re(\lambda_i) + j\,Im\{\lambda_i\})t} \cdot z_i(0)$$

$$z_i(t) = e^{Re(\lambda_i)t}\, e^{j\,Im(\lambda_i)t} \cdot z_i(0)$$

Note! $Re(\lambda_i)$ determines growth or decay rate with time

$Re(\lambda_i) < 0 \Rightarrow$ decay

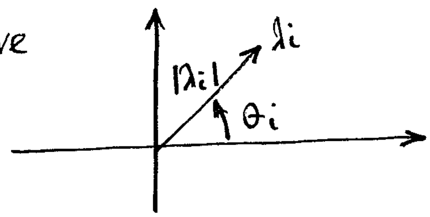
$Re(\lambda_i) > 0 \Rightarrow$ grow



Careful! Imaginary axis deserves more attention!

In DT: $\quad z_i(t) = \lambda_i^t z_i(0)$

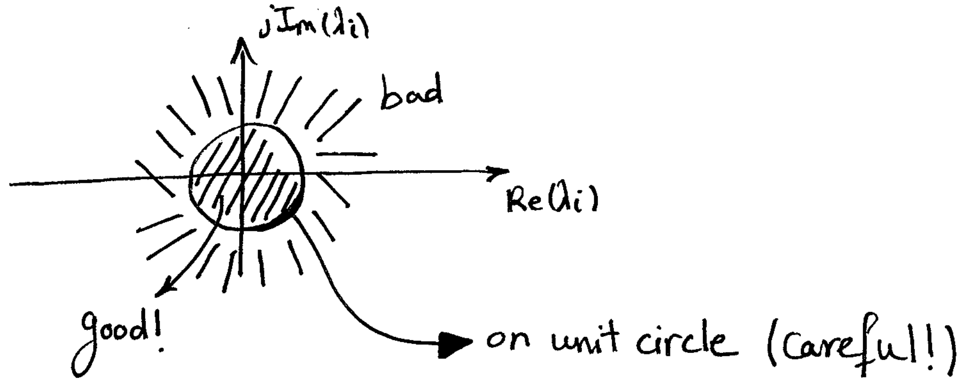
we have



In polar coordinates we have:

$$z_i(t) = (|\lambda_i| e^{j\theta_i})^t z_i(0)$$

$$\boxed{z_i(t) = |\lambda_i|^t e^{j\theta_i t} \cdot z_i(0)}$$



Recall: Double integrator $\quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$\lambda_1 = \lambda_2 = 0$

$$e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

In general for Jordan blocks:

$$J_3 = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \Rightarrow e^{J_3 t} = e^{\lambda t} \begin{bmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

$Re(\lambda) < 0 \Rightarrow$ asymptotic decay (as $t \to \infty$)

If $Re(\lambda) = 0 \Rightarrow$ response will blow up as $t \to \infty$.

Summary of Conditions for stability of LTI systems (unforced)

asymptotic decay in time

$\dot{x}(t) = Ax(t) \quad \dots \quad (*)$

(In terms of e-values of A)

System $(*)$ is

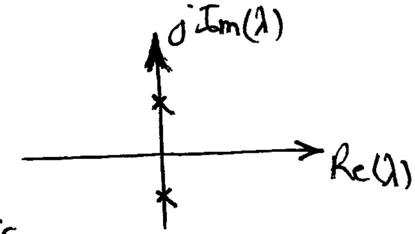
1. Stable $\Leftrightarrow Re(\lambda_i) < 0$
   $i = 1, \dots, n$

2. Unstable if either of these hold:

   (a) There is λ_i st. $Re(\lambda_i) > 0$

   (b) There is λ_i st. $Re(\lambda_i) = 0$ with algebraic multiplicity greater than or equal to 2.

Ex $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$



$\lambda_1 = j$
$\lambda_2 = -j$
algebraic multiplicity 1.

3. Marginally stable if all e-values have $Re(\lambda_i) \leq 0 \quad i = 1, \cdots, n$ and the e-values with $Re(\lambda_i) = 0$ have algebraic multiplicity one (no repeated e-values on $j\omega$-axis)

Ex $A = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}$

$\det(sI - A) = s^2 + a_1 s + a_0$

1) stable $\Leftrightarrow$ $a_1 > 0$, $a_0 > 0$ (Routh-Hurwitz criterion)

2) $a_1 = 0$

$\det(sI - A) = s^2 + a_0$

$a_0 < 0 \Rightarrow \det(sI - A) = s^2 - |a_0| \Rightarrow s_1 = \sqrt{|a_0|}$, $s_2 = -\sqrt{|a_0|}$ inverted pendulum

2(a) applying $\Rightarrow$ unstable

$a_0 = 0 \Rightarrow \det(SI - A) = S^2$

$S_1 = 0 \; ; \; S_2 = 0 \Rightarrow$ from 2(b) $\rightarrow$ unstable

$\hookrightarrow$ double integrator

$a_0 > 0 \Rightarrow S_1 = +j\sqrt{a_0}$

$S_2 = -j\sqrt{a_0}$ $\Rightarrow$ Case(3) Marginally stable

$\downarrow$

mass spring system

3) $a_0 = 0$

$\det(SI - A) = S^2 + a_1 S$

$$\left.\begin{array}{l} S_1 = 0 \\ S_2 = -a_1 \end{array}\right\} \begin{array}{l} \text{Marginally stable if } a_1 > 0 \\ \text{unstable if } a_1 \leq 0 \end{array}$$