Lecture 12

Linear Systems 10/16/12

last time:

Modal conditions for stability of

$\dot{x}(t) = A x(t)$ ... (*)

1) $Re(\lambda_i) < 0$ for all $i = 1, \ldots, n$

$\Rightarrow$ (*) stable

translation: all trajectories bounded and converge to zero as $t \to \infty$ (exponentially fast)

2) There is i st. $Re(\lambda_i) > 0$

or

$Re(\lambda_i) = 0$ w/ multiplicity ≥ 2

$\Rightarrow$ (*) unstable

translation: you can find $x(0)$ st. $x(t) = e^{At} x(0)$ becomes unbounded as $t \to \infty$

3) $Re(\lambda_i) < 0$ and there are simple e-values on $j\omega$-axis

$\Rightarrow$ (*) is marginally stable

<u>translation</u>: all trajectories are bounded for all time

* Stability of equilibrium points of

$\dot{x} = f(x) \quad \dots \quad (1)$

$\bar{x}$ is an e.p. of (1) if it solves $f(\bar{x}) = 0$

In linear case: $A\bar{x} = 0$ (we look for the nullspace of A)

$\bar{x} = 0$ unique e.p.

$\Updownarrow$

$\det(A) \neq 0$ (i.e. A doesn't have any e-values @ the origin)

If $\det A = 0 \Rightarrow Null(A)$ determines e-points of a linear system (there are ∞ many of them)

In nonlinear case (nonlinear systems) there are many options

1) $\bar{x}$ is a unique e.p.

Ex $\dot{x} = x^3$

$f(x) = x^3$

$f(\bar{x}) = 0 \Rightarrow \bar{x}^3 = 0 \Rightarrow \bar{x} = 0$ is a unique e.p.

2) Multiple e.p.

$\dot{x} = x(x-1)$

$f(x) = x(x-1) \Rightarrow f(\bar{x}) = 0 \Rightarrow \bar{x}(\bar{x}-1) = 0$

(2 e.q. points) $\bar{x}_1 = 0$
$\bar{x}_2 = 1$

3) Infinitely many

$\dot{x}_1 = x_2$
$\dot{x}_2 = -x_2^3$ $\Rightarrow f(\bar{x}_1, \bar{x}_2) = \begin{bmatrix} \bar{x}_2 \\ -\bar{x}_2^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$ $\bar{x}_1 \in \mathbb{R}$, $\bar{x}_2 = 0$

$\therefore$ e.p.'s are given by:

$(\bar{x}_1, \bar{x}_2) = (\text{anything}, 0)$

4) None

$\dot{x} = x^2 + 1$

$\bar{x}^2 + 1 = 0$ (over $\mathbb{R}$)

We'll assume that $\bar{x} = 0$ is an eq. point of $\dot{x} = f(x)$

Suppose that $\bar{x} \neq 0$ solves $f(\bar{x}) = 0$

Introduce a change of coordinates: $z = x - \bar{x}$

$(z(t) = x(t) - \bar{x})$

$$\boxed{\dot{z}(t) =} \dot{x}(t) - \dot{\bar{x}} = f(x) - \underbrace{f(\bar{x})}_{0} = f(x) \boxed{= f(z + \bar{x})}$$

In z-coordinates: $\dot{z} = f(z + \bar{x})$

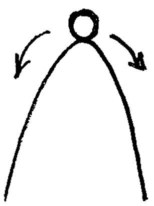
E.q. points solve: $f(\bar{z} + \bar{x}) = 0$

Note! Since $f(\bar{x}) = 0 \Rightarrow f(\underset{\bar{z}}{0} + \bar{x}) = 0$

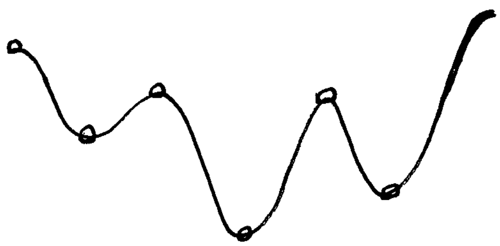
$\Rightarrow \bar{z} = 0$ is an e.q. point in z-coordinates.

* Initial condition perturbations away from e.p.

e.q. point $\longrightarrow$ this e.p. is asymptotically stable



$\longrightarrow$ e.q point unstable



no notion of global stability here.

$\bar{x}=0$ is (locally) asymptotically stable

$$\Updownarrow$$

1) for any $\varepsilon > 0$, there is a $\delta_1 > 0$ ($\delta_1 \leq \varepsilon$)

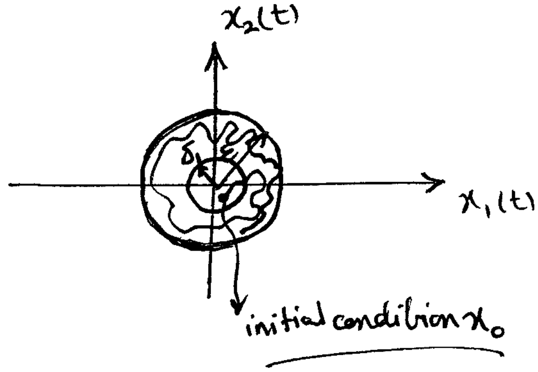
such that

$$\|x(0)\| < \delta_1 \Rightarrow \|x(t)\| < \varepsilon \quad \text{for all } t$$

$(\|x(0)-\bar{x}\|)$ $\qquad$ $(\|x(t)-\bar{x}\|)$

norm of a vector $x(0)$

Euclidean norm : $\|x(t)\| = \sqrt{x_1^2(t) + \cdots + x_n^2(t)}$

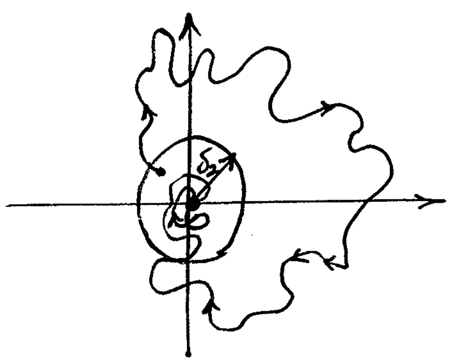


for all times if we start within the ball w/radius δ_1 we will never go out of the ball of radius ε

2) There is $\delta_2 > 0$ such that

$$\|x(0)\| < \delta_2 \quad \Rightarrow \quad \|x(t)\| \xrightarrow{t\to\infty} 0$$

$$\left(\lim_{t\to\infty} \|x(t)\| = 0\right)$$



Important!

(1) + (2) $\Rightarrow$ locally asymptotically stable

(1) + (2) (with $\delta_2 = +\infty$) $\Rightarrow$ globally asymptotically stable

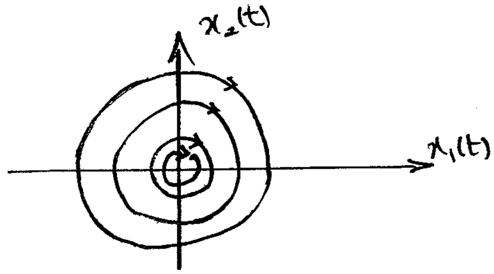
(1) ✓ but $\cancel{(2)}$ $\Rightarrow$ stable (in the sense of Lyapunov)

$\cancel{(1)}$ but (2) ✓ $\Rightarrow$ attractive

$\cancel{(1)}$ $\Rightarrow$ unstable

Ex1

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$



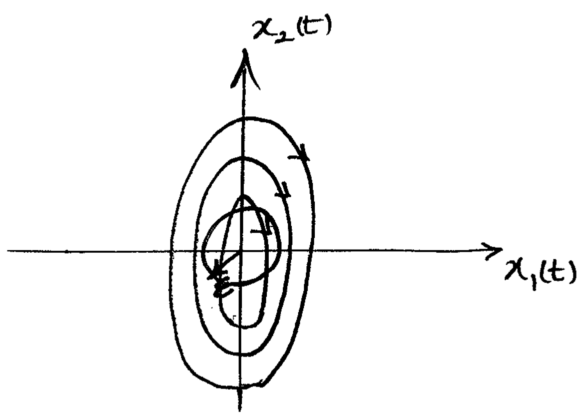
(1) ✓
(2) X

$\Rightarrow$ $\bar{x} = 0$ stable in the sense of Lyapunov
(but not asymptotically stable)

coorresponding linear system is marginally stable

Ex 2

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -k/m & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

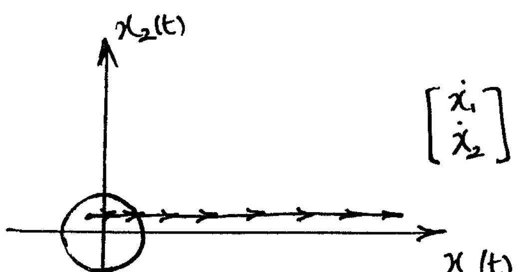


* determining direction of arrows:
$\dot{x}_1 = x_2$
$x_2 > 0 \Rightarrow x_1 \uparrow$

you can find ε st. by starting from the ε ball we will always stay inside the ball $\rightarrow$ if the ε ball lies across an ellipsoid holding x_0

Ex 3

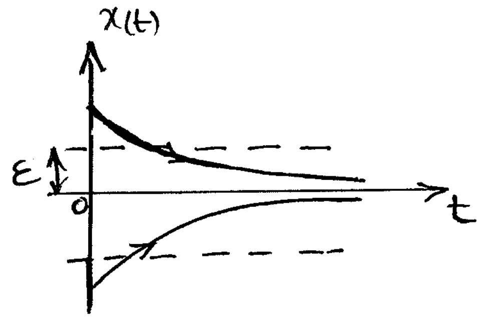
$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$



unstable

Ex 4 $\quad \dot{x} = -x^3 \quad ; \quad x > 0 \Rightarrow \dot{x} < 0 \Rightarrow x \downarrow$

$x < 0 \Rightarrow \dot{x} > 0 \Rightarrow x \uparrow$



locally & globally asymptotically stable