Lecture 13

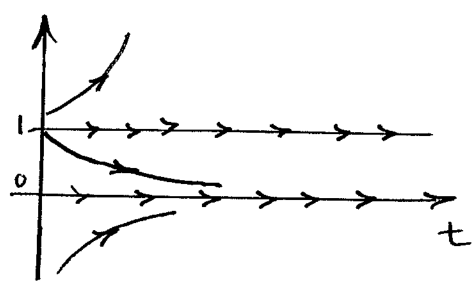
Linear Systems 10/18/12

Last time

- Stability of e.q. points of $\dot{x} = f(x)$

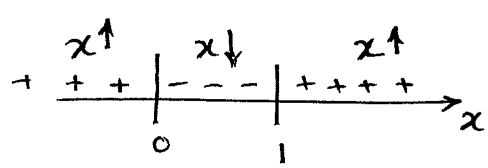
Ex $\dot{x} = x(x-1)$

e.p. : $\bar{x}(\bar{x}-1) = 0$ $\begin{cases} \bar{x}_1 = 0 \\ \bar{x}_1 = 1 \end{cases}$



$\dot{x} > 0 \iff x(x-1) > 0$

$(x(t) \uparrow)$



there are no glabally stable e.p. because we have two e.p's.

$\bar{x} = 0$ LAS

$\bar{x} = 1$ unstable

why is $\bar{x}=1$ not stable (in the sense of Lyapunov)

no matter how tight you choose boundaries (δ's) you will be moving farther away from $\bar{x}=1$ therefore there cannot be any notion of stability discussed around such an e.p.

Test for stability of e.p of nonlinear systems:

1) Linearization
(can be used to establish LAS or instability of e.p.)

we cannot discuss any notion of global stability when we are studying the linearized form of our nonlinear system around a certain eq. point.

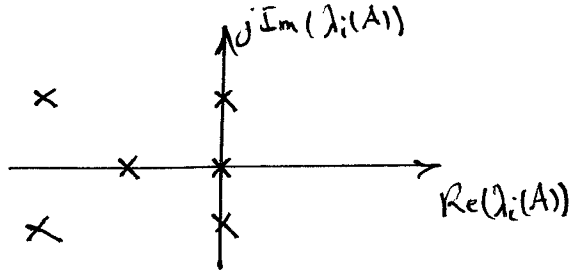
let $\bar{x}=0$ be an e.q. point of $\dot{x}=f(x)$; and let $\dot{\tilde{x}}=A\tilde{x}$ be a linearization of $\dot{x}=f(x)$ around $\bar{x}=0$ $\quad (A=\partial f/\partial x|_{x=\bar{x}})$
$\bar{x}=0$ is 8

1) LAS (locally asymptotically stable)

if $\dot{\tilde{x}}=A\tilde{x}$ is stable $(Re(\lambda_i(A))<0,\ i=1,\dots,n)$

2) Unstable if there is an e-value of A with a positive real part.
there is [illegible] st. $Re(\lambda_i(A))>0$

Note! further analysis is required if A has e-values on the $j\omega$-axis.



Ex

a) $\dot{x} = -x^3$

b) $\dot{x} = x^3$

$f(x) = \mp x^3$

$\left.\frac{\partial f}{\partial x}\right|_{\bar{x}} = \left.\mp 3x^2\right|_{\bar{x}=0} = 0$

So we will have :

a) $\dot{\tilde{x}} = 0\,\tilde{x}$

b) $\dot{\tilde{x}} = 0\,\tilde{x}$

since e-value was on $j\omega$-Axis we cannot conclude anything by doing this simple linearization.

Ex

$\dot{x} = x(x-1) = x^2 - x = f(x)$

$\frac{\partial f}{\partial x} = 2x - 1$

$\boxed{A_1} = \left.\frac{\partial f}{\partial x}\right|_{\bar{x}_1=0} = 2.0 - 1 = \boxed{-1}$

$$\boxed{A_2} \overset{!}{=} \left.\frac{\partial f}{\partial x}\right|_{\bar{x}_1 = 1} = 2 \cdot 1 - 1 \overset{!}{=} \boxed{+1}$$

$A_1 = -1 < 0 \Rightarrow \bar{x}_1 = 0$ is LAS.

$A_2 = +1 > 0 \Rightarrow \bar{x}_2 = 1$ is unstable.

Ex

$$\dot{x}_1 = -x_1 + 4x_2$$
$$\dot{x}_2 = -x_1 - x_2^3$$

$$0 = -\bar{x}_1 + 4\bar{x}_2 \Rightarrow \bar{x}_1 = 4\bar{x}_2$$
$$0 = -\bar{x}_1 - \bar{x}_2^3$$

$$\Rightarrow -4\bar{x}_2 - \bar{x}_2^3 = 0$$
$$\Rightarrow \bar{x}_2 (4 + \bar{x}_2^2) = 0 \Rightarrow \bar{x}_2 = 0$$

over the set of real numbers this is the only eq. point.

unique eq. point: $\bar{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

we can use linearization to study local stability of this point.

$$A = \begin{bmatrix} -1 & 4 \\ -1 & 0 \end{bmatrix} \qquad \left(\frac{\partial f}{\partial x} = \begin{pmatrix} -1 & 4 \\ -1 & -3x_2^2 \end{pmatrix} \right)$$

Routh Hurwitz criterion

$$\det(sI-A) = \det\begin{pmatrix} s+1 & -4 \\ 1 & s \end{pmatrix} = s^2+s+4 \Rightarrow \text{stable}$$

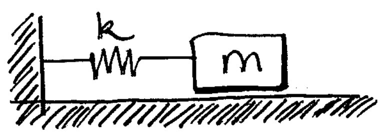
✓ LAS of $\bar{x}=0$ of original nonlinear system.

* It turns out that this point is also a globally stable point but nothing can be concluded at this point!

(with linearized system around this e.q point)

Ex (Intro to Lyapunov direct method)

mass spring system:



$$m\ddot{y}+ky=0$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\det(sI-A) = s^2+k/m \Rightarrow s_{1,2} = \pm j\sqrt{\tfrac{k}{m}}$$

$$y(t) = c_1 \sin\left(\sqrt{\tfrac{k}{m}}\,t\right) + c_2 \cos\left(\sqrt{\tfrac{k}{m}}\,t\right)$$

this system is marginally stable and oscillates forever if left by its own.

Energy : $E(t) = \underbrace{\frac{1}{2} k y^2(t)}_{\text{potential}} + \underbrace{\frac{1}{2} m \dot{y}^2(t)}_{\text{kinetic}}$

$= \frac{1}{2} k x_1^2(t) + \frac{1}{2} m x_2^2(t)$

$$\boxed{\frac{dE(t)}{dt}} \overset{!}{=} \cancel{\tfrac{2}{2}} k\, x_1(t)\dot{x}_1(t) + \cancel{\tfrac{2}{2}} m\, x_2(t)\dot{x}_2(t)$$

$$\overset{!}{=} \boxed{k\, x_1(t)\dot{x}_1(t) + m\, x_2(t) x_2}$$ holds in general

Objective : examine how energy changes along solutions of our system.

$\dot{x}_1 = x_2$

$\dot{x}_2 = -\frac{k}{m} x_1$

$$\boxed{\frac{dE(t)}{dt}} \overset{!}{=} k x_1 \cdot x_2 + \cancel{m} x_2 \left(-\frac{k}{\cancel{m}} x_1\right) = k x_1 x_2 - k x_1 x_2$$

$$\overset{!}{=} \boxed{0}$$

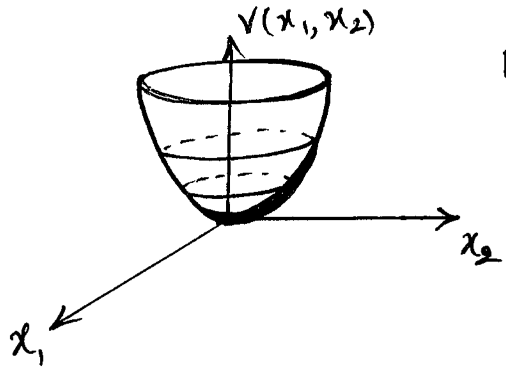
$\Rightarrow \forall t, \boxed{E(t)} \overset{!}{=} \text{const.} \overset{!}{=} \boxed{E(0)}$

energy of system doesn't increase or decrease.

$\Rightarrow$ marginally stable.

Lyapunov direct method

(read: a method for checking stability that doesn't rely on linearization or solution of the system)



Basic idea:

We want to study what is happening on level sets

$$V(x_1, x_2) = \text{const.}$$

Next time: we will impose certain <u>conditions</u> on a function $V(x)$: $V: \mathbb{R}^n \longrightarrow \mathbb{R}_+$

and we'll examine the sign of a derivative, $\frac{dV}{dt}$ along the solutions of $\dot{x} = f(x)$

$$\frac{dV}{dt} = \frac{\partial V}{\partial x} \frac{dx}{dt} = \frac{\partial V}{\partial x} \cdot f(x) \overset{?}{<} 0 \quad \text{check}$$

based on the sign of this derivative and the conditions imposed we will conclude certain results about the stability of the system.