Lecture 14 — linear Systems — 10/23/12

Direct method of Lyapunov

- A method for studying the stability of $\bar{x}=0$ that doesn't rely on:
  - linearization
  - finding the solution to $\dot{x}=f(x)$

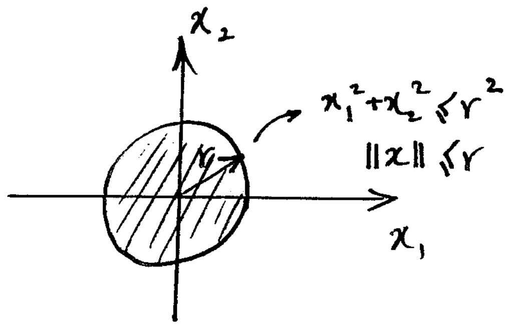
If $x(t)=\begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \in \mathbb{R}^n$ we'll be looking for scalar valued functions of the state

$$V: \mathbb{R}^n \longrightarrow \mathbb{R}$$

that have certain properties, and then we'll evaluate the derivative of this function w.r.t. time along the solutions of $\dot{x}=f(x)$.

A function $V: \mathbb{R}^n \longrightarrow \mathbb{R}$ is

1) locally positive definite if there is $r>0$ st. for all x with $\|x\| \leq r \Rightarrow \begin{array}{l} V(x)>0, \ x \neq 0 \\ V(0)=0 \end{array}$



Ex) in $\mathbb{R}^2$: $V(x) = x_1^2 + x_2^2$

2) globally positive definite

if (1) holds for $r = +\infty$

$\Rightarrow$ for all $x \in \mathbb{R}^n$ we have $\quad V(x) > 0 \ , \ x \neq 0$
$\quad V(0) = 0$

3) locally positive semi-definite if there is $r > 0$ st.

for all x with $\|x\| \leq r \Rightarrow \quad V(x) \geq 0 \ , \ x \neq 0$
$\quad V(0) = 0$

★ For negative definite properties, just flip the sign of $V(x)$.

Ex1 $\quad V(x) = a x_1^2 + b x_2^2$

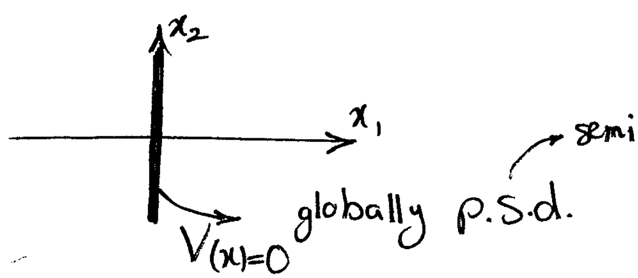
$$x(t) \in \mathbb{R}^2$$

equivalently we can write

$$V(x) = [x_1 \;\; x_2] \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

If $a > 0$, $b > 0$ $\Rightarrow$ $V(x)$ is globally p.d.

Q) $a > 0$, $b = 0$ $\Rightarrow$ $V(x) = a x_1^2$



Ex) $V(x) = x_1^2 + x_2^2$

$x(t) \in \mathbb{R}^2 \Rightarrow$ g.p.d.

$x(t) \in \mathbb{R}^3 \Rightarrow$ g.p.s.d.

$$V(x) = x_1^2 + x_2^2 + 0.x_3^2 = [x_1 \;\; x_2 \;\; x_3] \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Ex Quadratic Forms

Given $P = P^T$, $V(x) = x^T \underbrace{P x}_{y}$

Aside

$x^T P x$

$P \neq P^T$ but we can have $P = P_s + P_a$

$$= \frac{1}{2}(P + P^T) + \frac{1}{2}(P - P^T)$$

$$V(x) = x^T P x$$

$$= x^T P_s x + x^T P_a x$$

$$x^T P_a x = \frac{1}{2} x^T (P - P^T) x = \frac{1}{2} x^T P x - \frac{1}{2} x^T P^T x$$

$$= \frac{1}{2} (P^T x)^T x - \frac{1}{2} x^T \underbrace{(P^T x)}_{y}$$

$$\Rightarrow \boxed{x^T P_a x} = \frac{1}{2} y^T x - \frac{1}{2} x^T y$$

$$= \boxed{0}$$

$\rightarrow$ anti-symmetric part does not contribute to quadratic form.

so, if given a matrix P, in order to study definiteness of the quadratic form V(x) we can just take the symmetric part of P and disregard the anti-symmetric part.

$$V(x) = x^T P x = x^T P_s x$$

Fact! If matrix $P = P^T$ has all positive e-values

$$\lambda_1(P) > 0, \dots \lambda_n(P) > 0$$

$$\Downarrow$$

$$V(x) = x^T P x \text{ is g.p.d.}$$

Ex) $P = \begin{bmatrix} 1 & 1 \\ 3 & 5 \end{bmatrix}$, $P_s = \frac{1}{2}(P + P^T) = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$

$$V(x) = x^T P x = x^T P_s x = x_1^2 + 4x_1x_2 + 5x_2^2$$

If $P = P^T \rightarrow$ P is diagonalizable (unitarily diagonalizable)

$P = V \Lambda V^T$

$$V(x) = x^T V \Lambda \underbrace{V^T x}_{z} = z^T \Lambda z = \sum_{i=1}^{N} \lambda_i(P) z_i^2$$

z → a vector

$$\begin{bmatrix} z_1 \\ \vdots \\ z_N \end{bmatrix}^T \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_N \end{bmatrix} \Rightarrow$$

$\lambda_i(P) > 0 \Rightarrow$ p.d.

&

$\lambda_i(P) \geq 0 \Rightarrow$ p.s.d.

Alternative way of studying positive definiteness:

$P = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ we can study the principle minors

$\Delta_1 = 1$

$\Delta_2 = \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = 1 \cdot 5 - 2 \cdot 2 = 1$

det

★ Any positive definite function is a valid (legitimate) Lyap. function candidate for studying stability of $\bar{x} = 0$.

Once we find a Lyapunov function candidate we'll study $\frac{dV(x)}{dt}$

$$\frac{dV(x)}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} = \frac{\partial V}{\partial x}\cdot f(x)$$

Ex

$$V(x) = x_1^2 + x_2^2$$

$$\frac{\partial V}{\partial x} = \left[\frac{\partial V}{\partial x_1} \quad \frac{\partial V}{\partial x_2}\right] = \left[2x_1 \quad 2x_2\right]$$

$$\frac{dV(x)}{dt} = \left[2x_1 \quad 2x_2\right]\begin{bmatrix} f_1(x_1,x_2) \\ f_2(x_1,x_2) \end{bmatrix}$$

$$= 2x_1 f_1(x_1,x_2) + 2x_2 f_2(x_1,x_2)$$

* Given a locally positive definite function $V(x)$, the origin of $\dot{x} = f(x)$ is :

1) Stable in the sense of Lyapunov if $\frac{dV(x)}{dt}$ is locally negative semidefinite $\left(\dot{V}(x) \leq 0 \text{ forall } \|x\| \leq r,\ \dot{V}(0) = 0\right)$

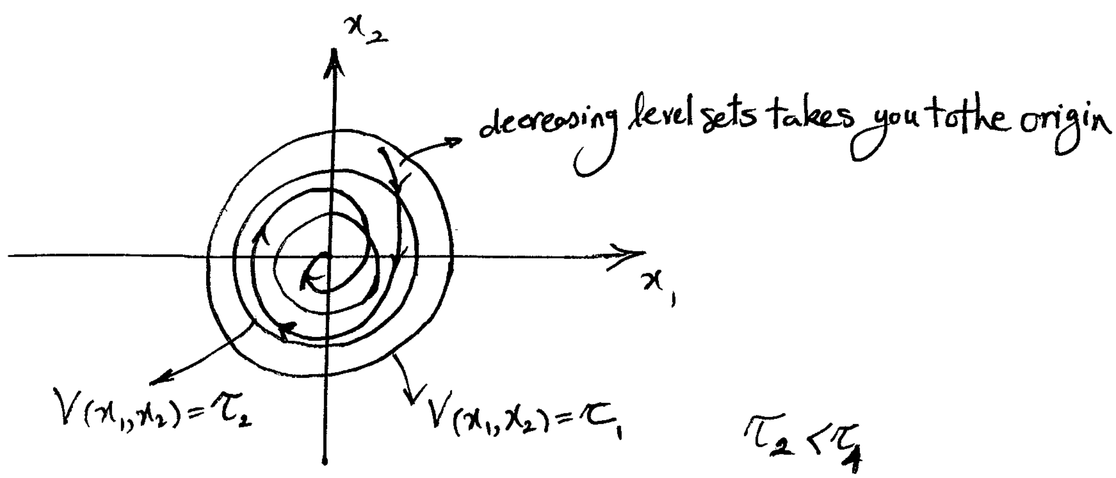
2) Locally Asymptotically stable if $\frac{dV(x)}{dt}$ is locally negative definite.

For global asymptotic stability we need

1) global positive definiteness of $V(x)$

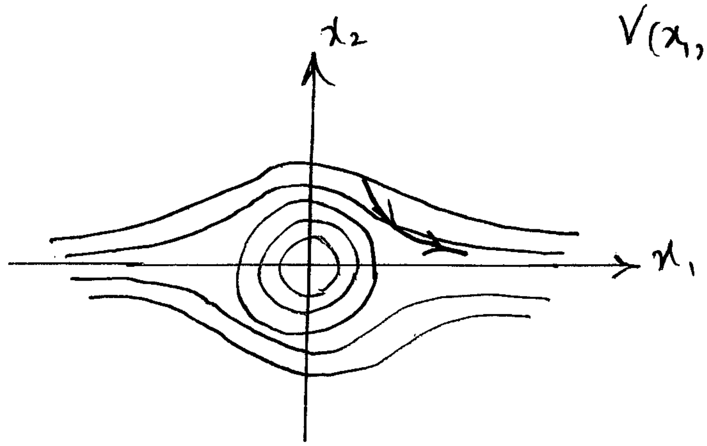
2) radial unboundedness of $V(x)$

3) global negative definiteness of $\frac{dV(x)}{dt}$



$\tau_2 < \tau_1$

$$V(x_1, x_2) = \frac{x_1^2}{1+x_1^2} + x_2^2$$



Radially unbounded $\Rightarrow \left(\|x\| \to +\infty \Rightarrow V(x) \to +\infty \right)$