Lecture 15 10/30/12

Linear Systems

last time:

Lyapunov direct method

$\bar{x} = 0$ of $\dot{x} = f(x)$ is GAS

1. $V(x)$ globally positive definite
2. $V(x)$ radially unbounded
   $\|x\| \to \infty \Rightarrow V(x) \to \infty$
3. $\dot{V}(x)$ globally negative definite

Back to linear systems

$\dot{x}(t) = Ax(t)$

We only need to look at quadratic Lyapunov functions.

$V(x) = x^T P x$ where $P = P^T > 0$

(P is symmetric positive definite matrix)

Positive definite matrix $\longleftarrow$ $\lambda_i(P) > 0$

$i = 1, \dots, n$

Q. Is $V(x)$ radially unbounded?

Fact

$$\boxed{\underbrace{\lambda_{min}(P)\ x^T x}_{\lambda_{min}(P)\|x\|_2^2} \leq x^T P x} \leq \underbrace{\lambda_{max}(P) \cdot x^T x}_{\lambda_{max}(P)\|x\|_2^2}$$

$$\lim_{\|x\| \to \infty} x^T P x \longrightarrow +\infty$$

To check stability, we need to examine the sign of $\dot{V}(x)$

$$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x} \overset{\dot{x} = Ax}{=\!=} (Ax)^T P x + x^T P (Ax)$$

$$= x^T A^T P x + x^T P A x$$

$$= x^T \underbrace{(A^T P + PA)}_{-Q} x = -x^T Q x$$

1) $\bar{x}=0$ is stable in the sense of Lyapunov,

($\dot{x}=Ax$ is marginally stable)

if $P=P^T>0$

$Q=Q^T\geq 0$

2) $\bar{x}=0$ is asymptotically stable ($\dot{x}=Ax$ is stable)

if $P=P^T>0$

$Q=Q^T>0$

<u>BIG Thm</u> :

$\dot{x}=Ax$ is stable

$\Updownarrow$ (iff)

For any $Q=Q^T>0$ there is a $P^T=P>0$ suchthat

$A^TP+PA=-Q$ ... (ALE, Algebraic Lyapunov Equation)

Moreover, the Lyapunov function is given by $V(x)=x^TPx$, where the unique solution of (ALE) is determined

by $P=\int_0^\infty e^{A^Tt}Qe^{At}dt$. $\blacksquare$

Always read the Lyapunov equation (ALE) from write to left !

$$A^TP + PA = -Q$$

Choosing arbitrary $P = P^T > 0$ does not guarantee that (ALE) is going to give $Q > 0$

★ $Q = Q^T > 0$ is arbitrary and then we look for $P = P^T > 0$.

Ex

$$A = \begin{bmatrix} -1 & 0 \\ K & -2 \end{bmatrix}$$

Lyapunov function candidate,

$$V(x) = x^Tx = x_1^2 + x_2^2$$

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

left hand side of (ALE)

$$A^TP + PA = \begin{bmatrix} -1 & K \\ 0 & -2 \end{bmatrix} P + P \begin{bmatrix} -1 & 0 \\ K & -2 \end{bmatrix}$$

$$A^TP + PA = A^T + A = \begin{bmatrix} -2 & K \\ K & -4 \end{bmatrix}$$

$$= -\begin{bmatrix} 2 & -K \\ -K & 4 \end{bmatrix}$$

$\begin{bmatrix} 2 & -K \\ -K & 4 \end{bmatrix} \overset{?}{>} 0$ check sign of principle minors

$\Delta_1 = 2 > 0$

$\Delta_2 = \det \begin{bmatrix} 2 & -K \\ -K & 4 \end{bmatrix} = 8 - K^2 > 0$

managed to show stability only if

$\Rightarrow K^2 < 8 \Rightarrow |K| < 2\sqrt{2}$

So this choise of the Lyapunov function lead us to this condition on K for the stability of the system.

Based on the position of the e-values of A we know that the system is stable! So P better be positive definite. From ALE we know that since A is stable and our choice of P is positive definite $\Rightarrow Q = Q^T > 0$

P is symmetric

Let's choose $Q = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ we have $P = \begin{bmatrix} P_1 & P_0 \\ P_0 & P_2 \end{bmatrix}$

$$PA = \begin{bmatrix} P_1 & P_0 \\ P_0 & P_2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ K & -2 \end{bmatrix} = \begin{bmatrix} -P_1 + KP_0 & -2P_0 \\ -P_0 + KP_2 & -2P_2 \end{bmatrix}$$

$$A^T P + PA = \begin{bmatrix} 2(-P_1 + KP_0) & * \\ -P_0 + KP_2 - 2P_0 & -4P_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$-2P_1 + 2KP_0 = -1 \Rightarrow P_1 = 1/2 + KP_0$$

$$-3P_0 + KP_2 = 0 \Rightarrow P_0 = K/3 \, P_2$$

$$-4P_2 = -1 \Rightarrow P_2 = 1/4$$

$$\Rightarrow P_2 = 1/4 \quad , \quad P_0 = K/12 \quad , \quad P_1 = 1/2 + \frac{K^2}{12}$$

Check positive definiteness of P!

$P > 0 \Rightarrow P_1 > 0$ and $P_1 P_2 - P_0^2 > 0$ ✓

So P is positive definite!

From the thm since A was stable any choice of $Q = Q^T > 0$ will result in a $P = P^T > 0$ as the solution of (ALE).

Sketch of the proof of the "BIG Thm" :

"$\Uparrow$" assume that for any $Q = Q^T > 0$ there is $P = P^T > 0$ s.t. $A^T P + PA = -Q$. Show $\dot{x} = Ax$ is stable.

Propose $V(x) = x^T P x$ as a Lyapunov function candidate.

$\Rightarrow$ g.p.d. radially unbounded.

$\dot{V}(x) = x^T (A^T P + PA) x = -x^T Q x$: g.n.d. (globally negative definite)

$\bar{x} = 0$ is GAS $\Rightarrow$ $\dot{x} = Ax$ is stable

"$\Downarrow$" assume $\dot{x}=Ax$ is stable

We'll show that $P=\int_0^\infty e^{A^T t} Q e^{At}\,dt$

satisfies :

1. $P=P^T$
2. $P>0$
3. ~~solves~~ (ALE)
4. Represents a unique sol'n to (ALE)

stability of A allows us to write e^{At} in the formation of P.

1. $$\boxed{P^T} = \int_0^\infty (e^{A^T t} Q e^{At})^T dt = \int_0^\infty e^{A^T t} \underbrace{Q^T}_{Q} e^{At}\,dt = \boxed{P}$$

2. $$x^T P x = \int x^T e^{A^T t} Q^{1/2} Q^{1/2} e^{At} x\,dt$$

$Q^{1/2}$ → sqrtm(Q) (Note! any positive definite matrix has a square root)

$$= \int_0^\infty z^T(t)\, z(t)\,dt = \int_0^\infty \|z(t)\|^2 dt = 0 \iff z(t)=0$$

$$z(t) = \underbrace{Q^{1/2}}_{\text{invertible}} \underbrace{e^{At}}_{\text{invertible}} x$$

$z(t) = 0$ only if $x = 0$.

so P is positive definite.