Lecture 17 11/13/12

Linear Systems

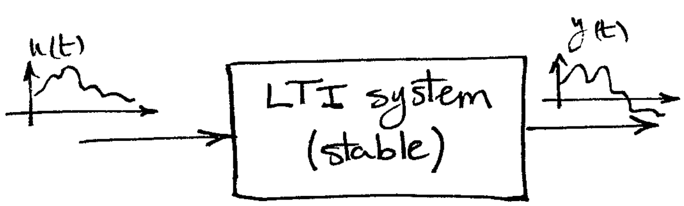
Last time:

- Frequency responses of SISO LTI systems

Today:

- Extensions to MIMO case

L_2-norm (energy of a signal)



$$\|u\|_2^2 = \int_0^\infty u^T(t)\,u(t)\,dt \quad , \quad y(t) = \int_0^t H(t-\tau)\,u(\tau)\,d\tau$$

$$\Updownarrow$$

$$Y(j\omega) = H(j\omega)\,U(j\omega)$$

$$\|y\|_2^2 = \int_0^\infty y^T(t)\,y(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y^*(j\omega)\,Y(j\omega)\,d\omega = \dots$$

parseval's equality

$$\cdots = \frac{1}{2\pi}\int_{-\infty}^{\infty} U^*(j\omega)\underbrace{H^*(j\omega)H(j\omega)}_{}U(j\omega)\,d\omega$$

At any ω this is a square Hermitian matrix (In SISO case $|H(j\omega)|^2$

Aside! $\begin{bmatrix}1+j\\2-j\\3+4j\end{bmatrix}^* = [1-j \mid 2+j \mid 3-4j]$ In Matlab: U'

If Matrix M is Hermitian $M^* = M$.

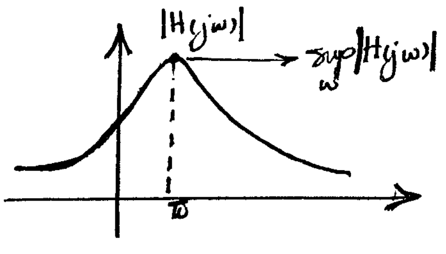
Ex. $(H^*H)^* = H^* \cdot (H^*)^* = H^*H$ so H^*H is Hermitian

SISO: $$\|y\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |H(j\omega)|^2 |U(j\omega)|^2 d\omega \leq \sup_\omega |H(j\omega)|^2 \cdot \frac{1}{2\pi}\int_{-\infty}^{+\infty} |U(j\omega)|^2 d\omega$$

$$= \sup_\omega |H(j\omega)|^2 \cdot \|u\|_2^2$$

$$\Rightarrow \quad \|y\|_2^2 \leq \sup_\omega |H(j\omega)|^2 \cdot \|u\|_2^2$$

L_2 norm of the output is upper bounded by the L_2 norm of the input times the $\underbrace{\text{largest value on the Bode mag. plot.}}_{\text{worst case amplification}}$



In fact, it can be shown that $\sup_{\omega} |H(j\omega)|^2$ is a tight upper bound

(There is a finite energy input s.t. $\|y\|_2^2 = \sup_{\omega} |H(j\omega)|^2 \|u\|_2^2$)

Conclusion: If we limit input signals to be of unit energy, then the worst case [largest] energy that the output can achieve is given by the square of the peak value on the Bode magnitude plot $\left(\sup_{\omega} |H(j\omega)|^2\right)$

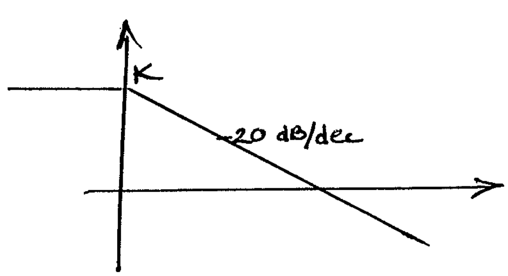
$$\sup_{0 < \|u\|_2^2 \leq 1} \frac{\|y\|_2^2}{\|u\|_2^2} = \underset{\text{finite input energy}}{\text{maximize}} \frac{\text{output energy}}{\text{input energy}} = \sup_{\omega} |H(j\omega)|^2$$

Ex

$$H(s) = \frac{K}{s+1}$$

$$H(j\omega) = \frac{K}{j\omega+1} \Rightarrow |H(j\omega)|^2 = \frac{K^2}{\omega^2+1}$$

$$\Rightarrow \sup_{\omega} |H(j\omega)|^2 = |H(j0)|^2 = K^2$$



Big Question : How to generalize this to MIMO case?

Answer : Proper generalization of $\sup_\omega |H(j\omega)|^2$ to MIMO case is given by $\sup \sigma_{max}(H(j\omega))$

where σ_{max} is the largest singular value of $H(j\omega)$.

A brief overview of SVD (Singular Value Decomposition)

What do we know? e-value decomposition of a square matrix $M \in \mathbb{C}^{n\times n}$

$Mv_i = \lambda_i v_i$. If M is diagonalizable, V is invertible.

then $MV = V\Lambda$

$V = [v_1 \cdots v_n]$, $\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$

If $M = M^*$ then $M = V\Lambda V^*$

V: matrix of e-vectors; Λ: diagonal of e-values

If a matrix is Hermitian it is unitarily diagonalizable.

What if $M \in \mathbb{C}^{m \times n}$ (rectangular matrix)

A square matrix U is unitary if $UU^* = U^*U = I \Rightarrow U^{-1} = U^*$

$$\|\underbrace{U \cdot x}_{\text{vector}}\|_{\underbrace{2}_{\text{eucledean norm}}}^2 = (U \cdot x)^*(U \cdot x) = x^* \underbrace{U^* U}_{I} x = \|x\|_2^2$$

Fact Any matrix $M \in \mathbb{C}^{m \times n}$ can be decomposed into

$$m\left\{\overbrace{\begin{bmatrix} M \end{bmatrix}}^{n}_{m \times n}\right. = \underset{m \times m}{\begin{bmatrix} U \end{bmatrix}} \underset{m \times n}{\begin{bmatrix} \Sigma \end{bmatrix}} \underset{n \times n}{\begin{bmatrix} V^* \end{bmatrix}}$$

$$\begin{bmatrix} \quad \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix} \begin{bmatrix} \quad \end{bmatrix} \begin{bmatrix} \quad \end{bmatrix}$$

where: $\left.\begin{array}{l} UU^* = U^*U = I_m \\ VV^* = V^*V = I_n \end{array}\right\} \Rightarrow U, V$ unitary

$$\Sigma = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r \end{bmatrix}$$

$\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \cdots \geq \sigma_r$

singular values of matrix M

$r = \text{rank}(M)$

Form MM^* and M^*M

$$\boxed{MM^*} = U\Sigma V^*(U\Sigma V^*)^* = U\Sigma \underbrace{V^*V}_{I}\Sigma^* U^*$$

$$= \boxed{U\Sigma\Sigma^* U^*}$$

$$\Rightarrow \quad (MM^*)U = U \cdot \Sigma\Sigma^*$$

$$[u_1 \cdots u_m] \qquad \begin{bmatrix} \sigma_1^2 & & & & & 0 \\ & \sigma_2^2 & & & & \\ & & \ddots & & & \\ & & & \sigma_r^2 & & \\ & 0 & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{bmatrix}$$

* thus e-value decomposition of a matrix MM^* gives left singular vectors u_i (columns of matrix U) and corresponding e-values determine $\sigma_1^2, \sigma_2^2, \ldots, \sigma_r^2$.

$$M^*M = V\Sigma^* \underbrace{U^* U}_{I} \Sigma V^* = V\Sigma^*\Sigma V^*$$

$$\Rightarrow \quad (M^*M) \cdot V = V \cdot (\Sigma^*\Sigma)$$

$$M^*M \cdot v_i = \sigma_i^2 v_i$$

v_i : right singular vectors

HW: Go to MIT notes,
read about SVD