Lecture 18 11/15/12

Linear systems

SVD of $M \in \mathbb{C}^{m\times n}$

Any matrix $M \in \mathbb{C}^{m\times n}$ can be represented as:

$$\underset{m\times n}{M} = \underset{m\times m}{U}\,\underset{m\times n}{\Sigma}\,\underset{n\times n}{V^*}$$

$$\left.\begin{array}{l} U \in \mathbb{C}^{m\times m} \\ V \in \mathbb{C}^{n\times n} \end{array}\right\} \Rightarrow \text{unitary} \qquad UU^* = U^*U = I$$

$$\Sigma = \left[\begin{array}{c|c} \begin{matrix} \sigma_1 & & \\ & \sigma_2 \ddots & \\ & & \sigma_r \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right]$$

$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r \geq 0$

$\hookrightarrow$ singular values (positive numbers)

$r = \text{rank}(M)$

$$MM^* u_i = \sigma_i^2 u_i \qquad i = 1, \ldots, m$$
$$M^*M v_i = \sigma_i^2 v_i \qquad i = 1, \ldots, n$$

$$\sigma_i^2 = \lambda_i(MM^*) = \lambda_i(M^*M)$$

$\lambda_i(\cdot)$: the e-values of a given matrix

$$U = [u_1 \; \cdots \; u_m]$$

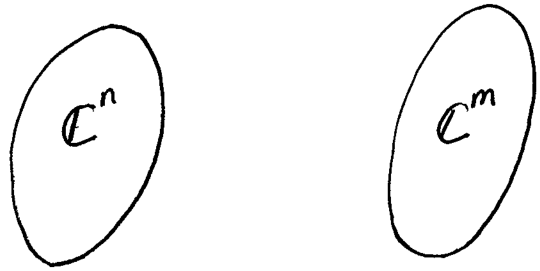
$$V = [v_1 \; \cdots \; v_n]$$

We'll think of matrices as mappings between $\underbrace{\text{linear spaces}}_{\text{(vector spaces)}}$.

$$M \in \mathbb{C}^{m \times n}$$

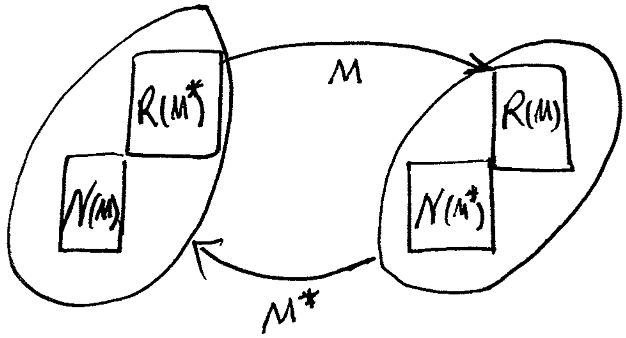
$$\underset{m\times 1}{\begin{bmatrix} g \end{bmatrix}} = \begin{bmatrix} & M & \end{bmatrix} \underset{n \times 1}{\begin{bmatrix} f \end{bmatrix}}$$

$$g = M \cdot f \Rightarrow M : \mathbb{C}^n \to \mathbb{C}^m$$



Range space of $M \longrightarrow R(M) = \{ g, \; g \in \mathbb{C}^m \text{ where } g = Mf, \; f \in \mathbb{C}^n \}$

Null space of $M \longrightarrow N(M) = \{ f, \; f \in \mathbb{C}^n \text{ s.t. } Mf = 0 \}$



$$\boxed{g} = M \cdot f = U \Sigma V^* f$$

$$= [u_1 \cdots u_m] \left[\begin{array}{c|c} \sigma_1 \cdots \sigma_r & 0 \\ \hline 0 & 0 \end{array}\right] \begin{bmatrix} v_1^* \\ \vdots \\ v_n^* \end{bmatrix} f$$

$$= \sum_{i=1}^{r} \underbrace{\sigma_i}_{\text{scalar}} u_i \underbrace{v_i^* f}_{\substack{\text{scalar} \\ c_i}} = \boxed{\sum_{i=1}^{r} c_i \sigma_i u_i}$$

Action of matrix M on vector f determined by a linear combination of $\{u_1, \ldots, u_r\}$

Thus: $\mathbf{R}(M) = \text{span}\{u_1, \ldots, u_r\}$

Note! We can write $f = \alpha_1 v_1 + \cdots + \alpha_n v_n$

$$g = \sum_{i=1}^{r} \sigma_i u_i v_i^* (\alpha_1 v_1 + \cdots + \alpha_n v_n)$$

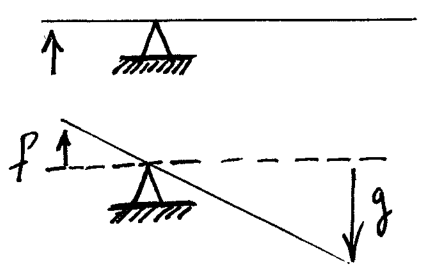
$\mathbf{v}_i^* \mathbf{v}_j = ?$

$$V^* V = \begin{bmatrix} v_1^* \\ v_2^* \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} v_1^* v_1 & v_1^* v_2 \\ v_2^* v_1 & v_2^* v_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$v_i^* v_j = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$g = (\alpha_1 \sigma_1 u_1 + \cdots + \alpha_r \sigma_r u_r) + \alpha_{r+1} \underbrace{\cancel{\sigma_{r+1}}}_{0} u_{r+1} + \cdots + \alpha_m \underbrace{\cancel{\sigma_m}}_{0} u_m$$

$$\mathcal{N}(M) = \{ v_{r+1}, \ldots, v_n \}$$



$$g = K \cdot f$$

How big is the amplification (gain) that can be achieved with matrix M ?

To answer this we need to decide how we measure the "size" of vectors f and g.

$$g = Mf$$

$$(\mathbb{C}^n, \|\cdot\|_2) \xrightarrow{M} (\mathbb{C}^m, \|\cdot\|_2)$$

$$\|f\|^2 = f^* f = |f_1|^2 + \cdots + |f_n|^2$$

$$\frac{\|g\|_2^2}{\|f\|_2^2} = \frac{\|Mf\|_2^2}{\|f\|_2^2}$$

$$\|Mf\|_2^2 = \|U\Sigma \underbrace{V^* f}_{h}\|_2^2 = \|U\Sigma h\|_2^2$$

$$= (U\Sigma h)^*(U\Sigma h) = h^* \Sigma^* \underbrace{U^* U}_{I} \Sigma h$$

$$= \sigma_1^2 |h_1|^2 + \cdots + \sigma_r^2 |h_r|^2$$

$$= \|\Sigma h\|_2^2$$

$$\|f\|_2^2 = \|V h\|_2^2 = \|h\|_2^2 \quad (V \text{ unitary matrix})$$

Now: we'll consider the worst case ratio of $\left.\frac{\|Mf\|_2^2}{\|f\|_2^2}\right|_{\text{s.t. } \|f\|_2^2 = 1}$

$$\sup_{\|f\|_2^2 = 1} \frac{\|Mf\|_2^2}{\|f\|_2^2} = \sup_{\|h\|_2^2 = 1} \|\Sigma h\|_2^2 = \sigma_1^2$$

$$\Rightarrow \quad \|Mf\|_2^2 = \|\Sigma h\|_2^2$$

Aside! $\quad \lambda_{min}(P)\,\|x\|_2^2 \leq x^T P x \leq \lambda_{max}(P)\,\|x\|_2^2$

σ_1 the maximum singular value $\longrightarrow$ 2-induced norm

Summary

$\frac{\|\cdot\|_2}{\|\cdot\|_2}$

2-induced gain of a matrix $M \in \mathbb{C}^{m \times n}$ is determined by the maximal (largest) singular value of M.

We are measuring size of vectors using Euclidean norm.

$$\|M\|_{2i} = \sup_{f \neq 0} \frac{\|Mf\|_2^2}{\|f\|_2^2} = \sigma_1^2(M)$$

induced

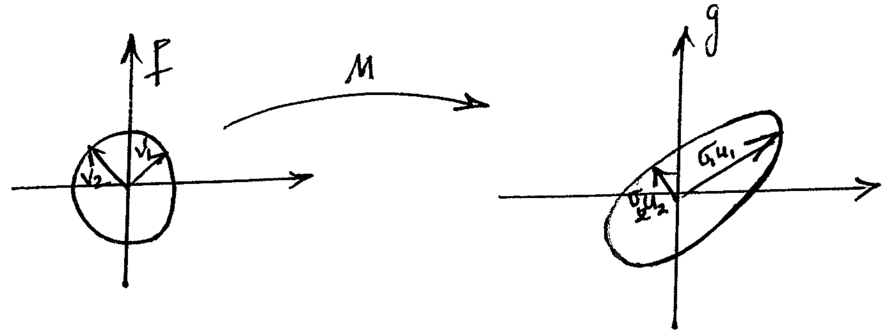
$$\frac{\|Mf\|_2^2}{\|f\|_2^2} \leq \|M\|_{2i}^2 \quad \Rightarrow \quad \|g\|_2^2 \leq \|M\|_{2i}^2 \cdot \|f\|_2^2$$

tight upper bound

$$\|M_1 M_2\|_{2i} \leq \|M_1\|_{2i} \|M_2\|_{2i}$$

submultiplicative property

Ex



$$f = \alpha_1 v_1 + \alpha_2 v_2$$

$$g = \alpha_1 \sigma_1 u_1 + \alpha_2 \sigma_2 u_2$$

put input in this direction

recall $g = Mf = \sum_{i=1}^{r} \sigma_i u_i v_i^* f \quad \underline{\underline{v_j}} \quad \sigma_j u_j$

an input in the direction of v_j results in a response in the direction of the corresponding singular vector u_j and amplification will be σ_j.