Lecture 19 11/16/12

Linear Systems

last time:

SVD of matrix $M \in \mathbb{C}^{m \times n}$

Today:

Relevance to systems theory

$M : \mathbb{C}^n \longrightarrow \mathbb{C}^m$

$$[g]_m = [\ M\ ]_{m \times n} [f]_n$$

using SVD: $g = \sum_{i=1}^{r} \sigma_i u_i v_i^* f$

$r = \text{rank}(M)$

$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$

Observations: $f = v_j \Rightarrow g = \begin{cases} \sigma_j u_j & j = 1, \ldots, r \\ 0 & j = r+1, \ldots, n \end{cases}$

$$\{v_{r+1}, \ldots, v_n\} = \mathcal{N}ull(M)$$
$$\hookrightarrow \mathcal{N}(M)$$

$$\{u_1, \ldots, u_r\} = Range(M)$$
$$\hookrightarrow \mathcal{R}(M)$$

characterize output (ie response obtained by the action of matrix M)

Recall what we had for diagonalizable matrices

$$Af = \sum_{i=1}^{n} \lambda_i v_i w_i^* f$$

$$A^* w_i = \bar{\lambda}_i w_i$$

$$A v_i = \lambda_i v_i$$

The 2-induced gain :

$$(\mathbb{C}^n, \|\cdot\|_2) \xrightarrow{M} (\mathbb{C}^m, \|\cdot\|_2)$$

$$\|M\|_{2i}^2 = \sup_{f \neq 0} \frac{\|Mf\|_2^2}{\|f\|_2^2} = \sigma_1^2(M)$$

largest singular value of M

Frequency response of MIMO system:

@ any fixed ω this is a (complex) matrix whose size is determined by the number of inputs and outputs

State-space :

$$\dot{x} = Ax + Bd \rightsquigarrow \text{input}$$
$$y = Cx + Dd$$

$$Y(s) = H(s) \cdot d(s)$$

$$H(s) = C(sI - A)^{-1}B + D$$

$$H(j\omega) = H(s)\Big|_{s=j\omega} = C(j\omega I - A)^{-1}B + D$$

For p : inputs
m : outputs

$$d(t) = \begin{bmatrix} d_1(t) \\ \vdots \\ d_p(t) \end{bmatrix}$$

$$y(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_m(t) \end{bmatrix}$$

$$H(j\omega) \in \mathbb{C}^{m \times p}$$

Ex

3 inputs
2 outputs
$\omega = 2$

$$\begin{bmatrix} Y_1(j\cdot 2) \\ Y_2(j\cdot 2) \end{bmatrix} = \begin{bmatrix} 2+j & 3-j & 2 \\ 3 & 0 & +j \end{bmatrix} \begin{bmatrix} U_1(j\cdot 2) \\ U_2(j\cdot 2) \\ U_3(j\cdot 3) \end{bmatrix}$$

Frequency response :

ω- parameterized family of matrices w/ complex entries

$$Y(j\omega) = \sum_{i=1}^{r} \sigma_i(j\omega)\, u_i(j\omega)\, v_i^*(j\omega)\, d(j\omega)$$

Physical interpretation of $H(j\omega)$

$$d(t) = d(j\omega)\, e^{j\omega t}$$

$$y(t) = Y(j\omega)\, e^{j\omega t}$$

Q: choose $d(j\omega)$: $\|d(j\omega)\|_2 = 1$ st. we have largest response [as measured by $\|\cdot\|_2$].

A: $d(j\omega) = v_1(j\omega)$

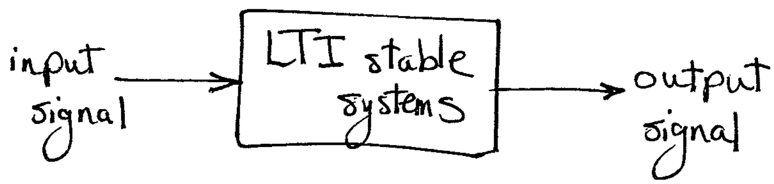
$$\Downarrow$$

$$Y(j\omega) = \sigma_1(j\omega)\, u_1(j\omega)$$

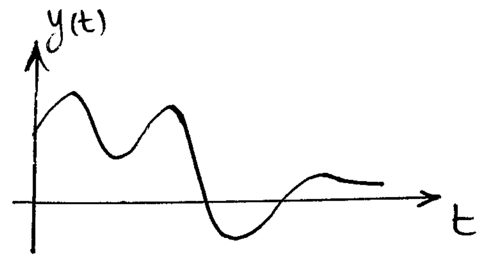
Fact: an L2-induced gain of a stable LTI system is determined by:

$$\sup_{\omega} \sigma_1(H(j\omega))$$

$$\|H\|^2_{L_2 \text{ induced}} = \, ?$$



L_2-norm of $y(t)$



$$\|y\|^2_{\substack{2 \\ \downarrow \\ L_2}} = \int_0^\infty \|y(t)\|_2^2 \, dt$$

$$= \int_0^\infty y^*(t)\, y(t) \, dt$$

So $$\|H\|^2_{L_2 \text{ induced}} = \sup_{0 \neq d \in L_2} \frac{\|y\|^2_{L_2}}{\|d\|_{L_2}} = \sup_{\omega} \bar{\sigma}_1^2 (H(j\omega))$$

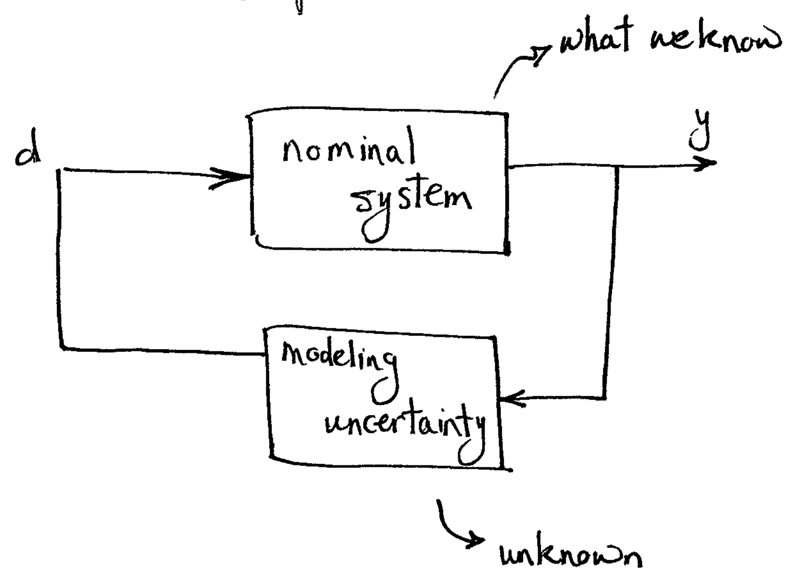
(for SISO systems $\sup_{\omega} |H(j\omega)|^2$)

* this is also known as the H_∞ norm of the system.

$\downarrow$

gives you worst case energy amplification

Robustness interpretation:



Aside! another useful matrix norm: Frobenius norm

$$M = \begin{bmatrix} 1+2j & 3-j & 0 \\ -1 & -2 & -3 \end{bmatrix} \Rightarrow \quad vec(M) = \begin{bmatrix} 1+2j \\ -1 \\ 3-j \\ -2 \\ 0 \\ -3 \end{bmatrix}$$

$$\boxed{\|M\|_F^2} = \|vec(M)\|_2^2 = (vec(M))^* vec(M) = trace(M^*M)$$

$$= trace(MM^*)$$

$$= \sum_i \lambda_i(MM^*)$$

$$= \boxed{\sum_{i=1}^{r} \sigma_i^2(M)}$$

Recall: for $A \in \mathbb{C}^{n \times n}$

$$\text{trace}(A) = \sum_{i=1}^{n} A_{ii} = \sum_{i=1}^{n} \lambda_i(A)$$

$$\| H(j\omega) \|_F^2 = \text{trace}\left(H(j\omega) H^*(j\omega) \right) = \sum_{i=1}^{r} \sigma_i^2 (H(j\omega))$$

power spectral density

$$\| H \|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace}\left(H(j\omega) H^*(j\omega) \right) d\omega$$

Parseval:

$$\| H \|_2^2 = \int_0^{+\infty} \text{trace}(\underbrace{H(t)}_{\text{impulse response}} H^*(t)) \, dt$$

$$\begin{cases} \dot{x} = Ax + Bd \\ y = Cx \end{cases}$$

$$H(t) = Ce^{At} B$$

$$\| H \|_2^2 = \int_0^{\infty} \text{trace}\left(Ce^{At} B B^* e^{A^* t} C^* \right) dt = \ldots$$

$$\cdots = \operatorname{trace}\left(C \int_0^\infty e^{At} BB^* e^{A^*t} dt \, C^*\right)$$

$$X_c = \int_0^\infty e^{At} BB^* e^{A^*t} dt$$

Controllability Grammian

$$AX_c + X_c A^* = -BB^*$$

Matlab: $X_c = \text{lyap}(A, BB^*) = \text{lyap}(A, BB')$

$$\|H\|_2^2 = \operatorname{trace}(C X_c C^*)$$

Alternatively:

$$\|H\|_2^2 = \int_0^\infty \operatorname{trace}(H^*(t) H(t)) \, dt$$

$$\|H\|_2^2 = \operatorname{trace}(B^* X_o B)$$

X_o: observability gramian

$$A^* X_o + X_o A = -C^* C$$

$$\|H\|_2^2 = \text{trace}\,(C X_c C^*) = \text{trace}\,(B^* X_o B)$$

L_1-norm : SISO $\displaystyle\int_0^\infty |h(t)|\,dt$