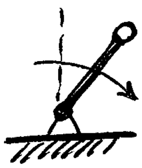
Lecture 20

Linear Systems 11/27/12

Reachability (Controllability)



DT: $x(k+1) = A\,x(k) + B u(k)$

LTI

solution: $x(k) = A^k x(0) + \sum_{i=0}^{k-1} A^{k-i-1} B u(i)$

Q. Given $x(0) = 0$, can we choose a sequence of inputs

$\{u(0), u(1), \dots, u(K_f)\}$

such that $x(K_f) = x_f \leftarrow$ given final state

(Reachability)

$$x_f = x(K_f) = \sum_{i=0}^{K_f-1} A^{K_f-i-1} B u(i) =$$

$$= A^{K_f-1} B u(0) + A^{K_f-2} B u(1) + \dots + B u(K_f-1)$$

$$= \underbrace{\left[A^{k_f-1}B \;\vdots\; A^{k_f-2}B \;\vdots\; \cdots \;\vdots\; B \right]}_{R_{k_f}} \underbrace{\begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(k_f-1) \end{bmatrix}}_{\underline{u}_{k_f}}$$

$x_f = R_{k_f} \cdot \underline{u}_{k_f}$ or for simplicity $x_k = R_k \underline{u}_k$

It is easy to see that $\underline{u}_k$ can be selected s.t.

$$x(k) = x_f \iff x_f \in \text{Range}(R_k)$$

$$R_{k+1} = \left[A^k B \;\vdots\; A^{k-1}B \;\vdots\; \cdots \;\vdots\; AB \;\vdots\; B \right] = \left[A^k B \;\vdots\; R_k \right]$$

$$R_{n+1} = \left[A^n B \;\vdots\; R_n \right]$$

By adding additional steps (columns) our ability to achieve certain stages grows. Is there a limit to this?

n: number of states in $x(k) = \begin{bmatrix} x_1(k) \\ \vdots \\ x_n(k) \end{bmatrix}$

* Cayley – Hamilton

Every square matrix $A \in \mathbb{R}^{n \times n}$ satisfies its own characteristic equation

$$f(s) = \det(sI - A) = s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0$$

$$A^n + a_{n-1} A^{n-1} + \cdots + a_1 A + a_0 I = 0$$

$$A^n = -a_{n-1} A^{n-1} - \cdots - a_1 A - a_0 I$$

Your reachability or ability to reach a certain state, saturates as the time step reaches n. So to summarize:

Reachability subspace $\mathcal{R}_k = \text{Range}(R_k)$

$$\mathcal{R}_k \subseteq \mathcal{R}_n = \mathcal{R}_l$$

$$k < n \leq l$$

i.e., if you can't reach given state in n steps you will never reach it.

System $x(k+1) = Ax(k) + Bu(k)$ is reachable

[Every $x \in \mathbb{R}^n$ can be reached in n or fewer time steps]

$$\Updownarrow$$

$$\text{rank}(R_n) = n$$

Kalman Rank Test 8

$$\text{rank}\left[A^{n-1}B \,\vdots\, A^{n-2}B \,\vdots\, \cdots \,\vdots\, AB \,\vdots\, B\right] = n$$

Note! For single input systems (1 control) reachability is equivalent to $\det R_n \neq 0$

Ex $n=2$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \cdot u(k)$$

$$R_2 = [AB \,\vdots\, B] \qquad AB = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 b_1 \\ \lambda_2 b_2 \end{bmatrix}$$

$$= \left[\begin{array}{c:c} \lambda_1 b_1 & b_1 \\ \lambda_2 b_2 & b_2 \end{array}\right]$$

$$\det(R_2) = \lambda_1 b_1 b_2 - \lambda_2 b_1 b_2 = b_1 b_2 (\lambda_1 - \lambda_2) \stackrel{?}{=} 0$$

System reachable $\Leftrightarrow$ $b_1 \neq 0$, $b_2 \neq 0$, $\lambda_1 \neq \lambda_2$

Non-reachable system

a) $b_2 = 0$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} b_1 \\ 0 \end{bmatrix} u(k)$$

$$x_1(k+1) = \lambda_1 x_1(k) + b_1 u(k)$$
$$x_2(k+1) = \lambda_2 x_2(k) + 0 . u(k)$$

$$\Rightarrow x_2(k) = \lambda_2^k x_2(0)$$

---

b) $\lambda_1 = \lambda_2 = \lambda$

$b_1 \neq 0 \quad b_2 \neq 0$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u(k)$$

$R_2 = \left[\begin{array}{c|c} \lambda b_1 & b_1 \\ \lambda b_2 & b_2 \end{array}\right]$ is a singular matrix

$\text{Range}(R_2) = \left\{ \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right\}$ the only things we can reach are $\alpha \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$\text{rank}(R_2) = 1$

k-step Reachability Gramian

$$P_k = R_k R_k^T = \sum_{i=0}^{k-1} A^i B B^T (A^i)^T$$

(CT: $P_t = \int_0^t e^{A\tau} B B^T e^{A^T \tau} d\tau$

$$A P_k A^T = \sum_{i=0}^{k-1} A^{\overset{l}{(i+1)}} B B^T (A^{i+1})^T = \sum_{l=1}^{k} A^l B B^T (A^l)^T$$

$$= \sum_{l=1}^{k} A^l B B^T (A^l)^T + A^0 B B^T (A^0)^T$$

$$- A^0 B B^T (A^0)^T$$

$\Rightarrow \quad A P_k A^T - P_{k+1} = -B B^T$ (Lyapunov equation)

$$\Rightarrow \quad \boxed{P_{k+1} = A P_k A^T + B B^T}$$

$$P_0 = B B^T$$

As $k \to \infty$ and limits exist ($|\lambda_i(A)| < 1 \quad i = 1, \ldots, n$)

$$\lim_{k \to \infty} P_k = P_\infty \quad \Rightarrow \quad A P_\infty A^T - P_\infty = -B B^T$$

Fact:

$$\text{Range}(P_k) = \text{Range}(R_k)$$

* Minimum energy problem

$$\left.\begin{array}{l} x_f = R_k \underline{u}_k \\ x_f \in \text{Range}(R_k) \end{array}\right\} \quad \text{there is } \underline{u}_k \text{ s.t. } x_f = R_k \underline{u}_k$$

Min. energy : solve the following problem

$$\begin{array}{ll} \text{minimize} & \underline{u}_k^T \underline{u}_k \\ \text{s.t.} & R_k \underline{u}_k - x_f = 0 \end{array}$$