Lecture 21

Linear Systems 11/29/12

Last time:

- Reachability of DT LTI systems

- Kalman rank test

Today:

- Reachability Gramian
- Minimum energy control
- Standard form for unreachable systems

Recall: $x(0)=0$ $\oplus$ $x(k+1) = Ax(k) + Bu(k)$

$$x(k) = \left[A^{k-1}B \;\vdots\; \cdots \;\vdots\; B \right] \begin{bmatrix} u(0) \\ \vdots \\ u(k-1) \end{bmatrix} = R_k \underline{u}_k$$

$x(k+1) = Ax(k) + Bu(k)$ (*)

↳ Reachability ✓ [i.e. we can reach any $x_f \in \mathbb{R}^n$ in n (or fewer) time steps]

$\Longleftrightarrow \quad \operatorname{rank} R_n = n$

$$R_n = [A^{n-1}B \mid A^{n-2}B \mid \dots \mid AB \mid B]$$

on-off test

Reachability Gramian : $\quad P_k = R_k R_k^T = \sum_{i=0}^{k-1} A^i B B^T (A^i)^T$

Lyapunov Eq :

$A P_k A^T - P_{k+1} = -BB^T$ $\quad$ for stable systems $(|\lambda_i(A)| < 1)$

$A P_\infty A^T - P_\infty = -BB^T$

$P_\infty = \lim_{k \to \infty} P_k$

CT: $\operatorname{lyap}(A, B) = P_\infty$

DT: $\operatorname{dlyap}(A, B) = P_\infty$ check

$P_k \geq 0$ (positive semi definite)

$$x^T P_x x = x^T R_k \underbrace{R_k^T x}_{z} = z^T z = \sum z_i^2 \geq 0$$

$$P_\ell - P_k \geq 0 \quad , \ \ell \geq k$$

$$P_\ell = \sum_{i=0}^{\ell-1} A^i B B^T (A^i)^T = P_k + \underbrace{\sum_{i=k}^{\ell-1} A^i B B^T (A^i)^T}_{(*)}$$

$$x^T P_\ell x = x^T P_k x + \underset{\geq 0}{(*)}$$

Assume:

- Reachability of $*$
- Let $k > n$

$$x_f = R_k \underline{u}_k$$

$$\underbrace{\begin{bmatrix} x_f \end{bmatrix}}_{n} = \underbrace{\begin{bmatrix} \quad & \quad & \quad \end{bmatrix}}_{\substack{\text{fat matrix} \\ n \times m \cdot k}} \underbrace{\begin{bmatrix} \underline{u}_k \end{bmatrix}}_{k \cdot m}$$

$m \rightarrow$ # of controls

$k \rightarrow$ # of time steps

There are many control inputs that can bring us to $x(k) = x_f$

Choose one with the minimum energy

Minimum energy state transfer (from $x(0) = 0$ to $x(K) = x_f$)

$$
\begin{cases}
\text{minimize} & \underline{u}_k^T \underline{u}_k \\
\text{subject to} & x_f - R_k \underline{u}_k = 0
\end{cases}
$$

Problem data : x_f , R_k

Optimization variable : $\underline{u}_k = \begin{bmatrix} u(0) \\ \vdots \\ u(k-1) \end{bmatrix}$
(unknown)

$$x_f \longleftarrow \boxed{R_k} \xleftarrow{\;\underline{u}_k\;}$$

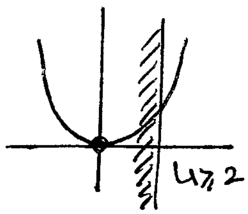
Plant (system we'd like to control)

For the minimization above : in the absence of problem constraints the optimal value is zero which is achieved by $\underline{u}_k = 0$.

$$\underline{u}_k^T \cdot \underline{u}_k = \sum_{i=0}^{k-1} \underbrace{u^T(i)\, u(i)}_{\text{quadratic} \geq 0} \geq 0$$

If minimize u^2

$$(u^2)' = 2u = 0$$



form lagrangian :

$$\mathbf{L}(\underline{u}_k, \lambda) = \underline{u}_k^T \underline{u}_k + \lambda^T (x_f - R_k \underline{u}_k)$$

Lagrange multiplier

(Price for violating constraints)

$$\frac{\partial L}{\partial \underline{u}_k} = 2\underline{u}_k^T - \lambda^T R_k = 0$$

$$\frac{\partial L}{\partial \lambda} = x_f - R_k \underline{u}_k = 0 \qquad \underline{u}_k = \tfrac{1}{2} R_k^T \lambda$$

$$\Rightarrow \underbrace{x_f}_{\text{known}} = \tfrac{1}{2} \underbrace{R_k R_k^T}_{\text{known}} \underbrace{\lambda}_{\text{unknown}}$$

Aside

R_k full rank matrix

$\Downarrow$

$R_k R_k^T$ invertible

$$\Rightarrow \lambda = 2(R_k R_k^T)^{-1} x_f$$

$$\underline{u}_{opt} = \underbrace{R_k^T (R_k R_k^T)^{-1}}_{Pinv(R_k)} x_f$$

pseudo_inverse $\longrightarrow$ Pinv(R_k)

$$x_f \longleftarrow \boxed{R_k} \xleftarrow{\;\underline{u}_{opt}\;} \boxed{R_k^T (R_k R_k^T)^{-1}} \longleftarrow x_f$$

$$\underline{u}_{opt} = R_k^T P_k^{-1} x_f$$

Note! If the system is not reachable we can add $(x_f - R_k \underline{u}_k)^T (x_f - R_k \underline{u}_k)$ to the objective function which is a least squares problem.

$$\boxed{\underline{u}_{opt}^T \underline{u}_{opt}} = x_f^T P_k^{-1} \underbrace{R_k R_k^T}_{P_k} P_k^{-1} x_f = \boxed{x_f^T P_k^{-1} x_f}$$

$$P_l \geq P_k \, , \quad l \geq k$$

$$\Rightarrow \; P_k^{-1} \geq P_l^{-1} \, , \quad l \geq k$$

Summary! The longer we wait, the less energy we spend.

$$x_f = R_k \underline{u}_k \overset{\text{SVD}}{=} U\Sigma V^* \underline{u}_k$$

$$= \sum \sigma_i u_i v_i^* \cdot \underline{u}_k$$

If we choose input $\underline{u}_k = u_j \Rightarrow x_f = \sigma_j u_j$

(from SVD) $U = [u_1 \cdots u_n]$

$$P_k = R_k R_k^T = U\Sigma \underbrace{V^* V}_{I} \Sigma^* U^*$$

$$P_k U = U \Sigma \Sigma^*$$

$$P_k u_i = \sigma_i^2 u_i \longrightarrow$$ e-value decomposition of P_k

$$P_k^{-1} u_i = \frac{1}{\sigma_i^2} u_i$$

Q. If we want to have $x_f = u_i$, what is $\underline{u}_{opt}^T \underline{u}_{opt}$?

$$\boxed{\underline{u}_{opt}^T \underline{u}_{opt}} \overset{!}{=} x_f^T P_k^{-1} x_f = u_i^T P_k^{-1} u_i = u_i^T \left(\frac{1}{\sigma_i^2}\right) u_i \quad (u_i\text{'s orthonormal})$$

$$= \boxed{\frac{1}{\sigma_i^2}}$$

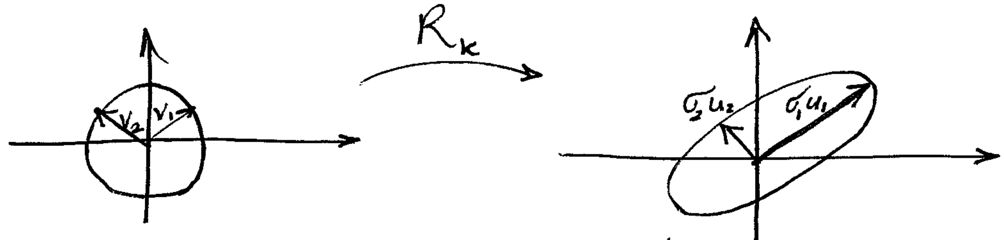
By using $x_f = u_i \rightarrow$ energy is proportional to the inverse of the corresponding e-value of the reachability Gramian

Summary! evalue decomposition of the Reachability Gramian provides insight into how easy it is to achieve some final state.

[e-values small $\Rightarrow$ direction difficult]

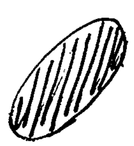
[e-values large $\Rightarrow$ easy]

Space of control:



reachability ellipsoide which grows with time, but will not grow beyond certain level.

with unit energy input all points inside this ellipsoid can be reached



Reachability ellipsoid tells us what we can reach w/ unit energy input.