Lecture 22 11/30/12

Linear Systems

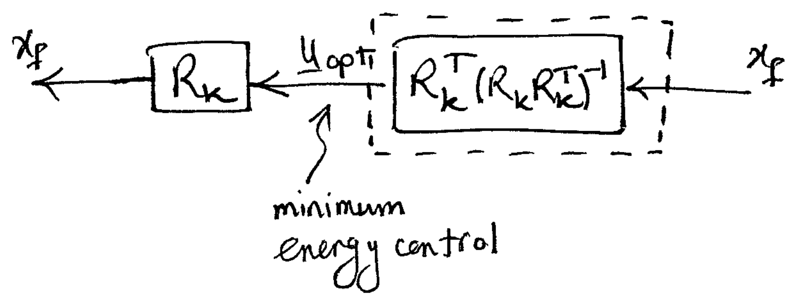
Last time

Minimum energy state transfer
Reachability ellipsoid

Key: Reachability Gramian

Today many things!

- Canonical forms of unreachable systems
- Modal Tests for reachability
- CT
- Observability
- Pole Placement
- Observer design



$P_k = R_k R_k^T$: Provides quantitative measure of how easy some directions are to achieve

$P_k u_i = \sigma_i^2 u_i$

$$x_f = [\quad] \begin{bmatrix} u(0) \\ \vdots \\ u(k-1) \end{bmatrix}$$

Fact: $R_n = [A^{n-1}B \mid \cdots \mid AB \mid B]$

$\text{Range}(R_n) =$ column span of $\{A^{n-1}B, \ldots, B\}$

$$AR_n = [A^n B \mid \cdots \mid AB]$$

$-(a_{n-1}A^{n-1} \ldots - a_0 I)$ Cayley-Hamilton

Conclusion: $\text{Range}(R_n)$ is A-invariant ($z \in \text{Range}(R_n) \Rightarrow Az \in \text{Range}(R_n)$)

Canonical Form for unreachable systems :

Let $\text{rank}(R_n) = r < n$ ← # of states

$$x(k+1) = A\,x(k) + B\,u(k)$$

Introduce change of variables

$$x(k) = T\,z(k)$$

$$T\,z(k+1) = A\,T\,z(k) + B\,u(k)$$

$$z(k+1) = \bar{A}\,z(k) + \bar{B}\,u(k)$$

$$\bar{A} = T^{-1}AT \quad \Big| \quad \bar{B} = T^{-1}B$$

$$\mathbf{T} = [\,T_1 \,\vdots\, T_2\,]$$

Choose T_1 s.t. its columns span Range(R_n)

Choose T_2 to gain invertibility of T.

→ columns mutually (linearly) independent and linearly indep. of columns of T_1

$$\underbrace{R_n}_{\text{SVD}} = U\Sigma V^* = \sum_{i=1}^{r} \sigma_i u_i v_i^*$$

$$\text{Range}(R_n) = \{u_1, \dots, u_r\}$$

$T_1 = [u_1 \vdots \cdots \vdots u_r]_{n\times r}$ one choice for $T_2 = [u_{r+1} \vdots \cdots \vdots u_n]_{n\times(n-r)}$

$$\bar{A} = T^{-1} A T \quad ; \quad \bar{B} = T^{-1} B$$

$$T\bar{A} = AT \quad ; \quad T\bar{B} = B$$

$$[T_1 \mid T_2] \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} = A [T_1 \mid T_2]$$

$$[T_1 \bar{A}_{11} + \underbrace{\cancel{T_2 \bar{A}_{21}}}_{0} \mid T_1 \bar{A}_{12} + T_2 \bar{A}_{22}] = [\underbrace{AT_1}_{\text{span}\{u_1, \cdots, u_r\}} \mid AT_2]$$

Thus $\bar{A}$ has to be of this form $\bar{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix}$

$$[T_1 \;\; T_2] \bar{B} = B$$

$$[T_1 \;\; T_2] \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} = B \quad \Rightarrow \quad T_1 \bar{B}_1 + \underbrace{\cancel{T_2 \bar{B}_2}}_{0} = B$$

<u>Conclusion</u>

$$\begin{bmatrix} z_1(k+1) \\ z_2(k+1) \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} + \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix} u(k)$$

$z_1(k) \in \mathbb{R}^r$ reachable state

$z_2(k) \in \mathbb{R}^{n-r}$ unreachable part of state

$$z_2(k+1) = \bar{A}_{22} z_2(k) + 0.u(k)$$

$$z_2(k) = \bar{A}_{22}^{k} z_2(0)$$

no matter what we choose for our control input, z_2 is unreachable (z_2 evolves on its own independently of u)

★ Modal conditions for rechability

Thm System $x(k+1) = Ax(k) + Bu(k)$ is unreachable

$\Updownarrow$ (equivalent to)

There is a left e-vector of A: $w^T A = \lambda w^T$ s.t. $w^T B = 0$.

proof "$\Leftarrow$" Assume $w^T A = \lambda w^T$, $w^T B = 0$ & show lack of reachability.

$$W^T B = 0$$

$$W^T A B = \lambda W^T B = 0$$

$$W^T A^{n-1} B = 0$$

$$W^T \underbrace{[A^{n-1}B \mid \cdots \mid B]}_{R_n} = 0$$

but you cannot multiply a full rank matrix by a matrix from the left and get zero.

$\Rightarrow R_n$ is not full rank

~~unreachable~~ ✓

" $\Rightarrow$ "

Assume the system is not reachable.

Therefore we can bring it into the canonical form discussed earlier, i.e,

$$\begin{bmatrix} z_1(k+1) \\ z_2(k+1) \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} + \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix} u(k) .$$

Consider $\qquad W_2^T \bar{A}_{22} = \lambda W_2^T$

Fact:
$$[0 \quad W_2^T] \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix} = [0 \quad W_2^T \bar{A}_{22}] = [0 \quad \lambda W_2^T]$$

$$= \lambda [0 \quad W_2^T]$$

but $\boxed{\begin{bmatrix} 0 & W_2^T \end{bmatrix} \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix} = 0\bar{B}_1 + W_2^T 0 = 0}$

PBH Test : (Modal test for reachability or lack of it)

$x(k+1) = Ax(k) + Bu(k)$ reachable

$$\Updownarrow$$

$$\operatorname{rank}([zI - A \mid B]) = n$$

for all $z \in \mathbb{C}$

Note! We only need to check e-values of A, i.e.,

$$z \in \{\lambda_1(A), \dots, \lambda_n(A)\}$$

Sketch of proof: $\underset{z=\lambda}{W^T[zI-A \mid B]} \Rightarrow W^T[\lambda I - A \mid B] =$

$$= [\lambda W^T - W^T A \mid W^T B]$$

$$= [0 \mid 0]$$

Ex

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k)$$

$$\lambda I - A = \begin{bmatrix} \lambda - 1 & -3 \\ 0 & \lambda - 2 \end{bmatrix}$$

check $\text{rank}\left(\left[\begin{array}{cc|c} \lambda_i - 1 & -3 & 1 \\ 0 & \lambda_i - 2 & \underbrace{0}_{B} \end{array} \right] \right)$ for $\lambda_1 = 1$, $\lambda_2 = 2$

for λ_1: $\text{rank}\left(\left[\begin{array}{cc|c} 0 & -3 & 1 \\ 0 & -1 & 0 \end{array} \right] \right) = 2$

for λ_2: $\text{rank}\left(\begin{bmatrix} 0 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right) = 1$

$\Rightarrow$ λ_1 is reachable but λ_2 is not.