Lecture 23 11/30/12

Linear Systems

Controllability

Transfer non-zero initial state to zero

$x(k+1) = A\,x(k) + Bu(k)$

$\cancel{x(k)}^{\,0} = A^k x(0) + R_k \underline{u}_k$

$-A^k x(0) = R_k \underline{u}_k$

Continuous time system:

$\dot{x}(t) = A\,x(t) + B\,u(t)$

$$x(t) = e^{At} x_0 + \int_0^t e^{A(t-\tau)} B\,u(\tau)\,d\tau$$

Note! Reachability $\Longleftrightarrow$ Controlability

e^{At}: invertible

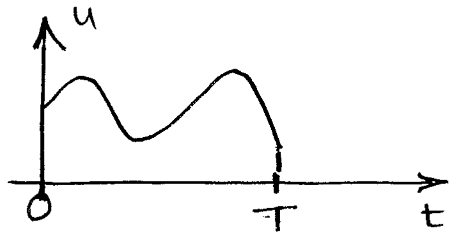
$x_0 = 0$ and study reachability

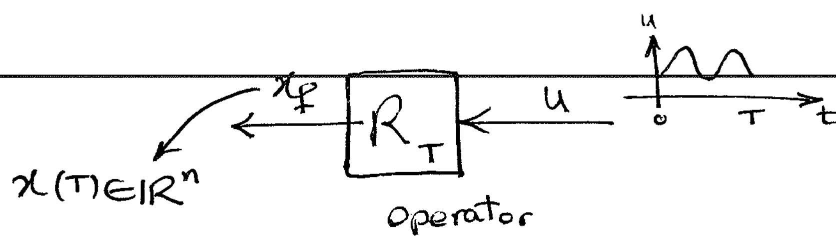
$[0, T]$

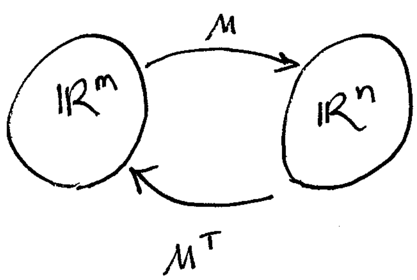
$$x(T) = x_f = \int_0^T e^{A(T-\tau)} B u(\tau)\, d\tau = [R_T u](T)$$

Control $u[0,T]$

$\rightsquigarrow$ function on interval $[0,T)$







Everything holds for CT from DT by replacing transposes by its adjoint.

Fact: Minimum energy control

$$x_f \leftarrow \boxed{R_T} \xleftarrow{u_{opt}} \boxed{R_T^{ad}(R_T R_T^{ad})^{-1}} \leftarrow x_f$$

$$P_T = \int_0^T e^{At} B B^T e^{A^T t}\, dt = R_T R_T^{ad}$$

$$\frac{dP_T}{dT} = AP + PA^T + BB^T$$

Fact 1 : Controlability in CT is not influenced by the choice of T (final time)

Fact 2 : Kalman rank test holds:

System reachable (controllable)

$$\Updownarrow$$

$$\text{rank}([A^{n-1}B \mid \cdots \mid B]) = n$$

Fact 3 : PBH test and Kalman Canonical forms for unreachable systems carry over from DT.

$$\text{rank}([sI - A \mid B]) = n$$

$$u_{opt}(t) = \underbrace{B e^{A^T t}}_{R_T^{ad}} \cdot P_T^{-1} \cdot x_f$$

Controllability gramian at time T

Observability

Determines our ability to estimate states from limited measurements.

DT: $$x(k+1) = A\,x(k)$$
$$y(k) = C\,x(k)$$

A, C : model of our system (given matrices)

y : measured output

x : state [we want to estimate]

$$x(k) = A^k \cdot x(0)$$

so your ability to see the state at stage k depends on your ability to see $x(0)$ with limited measurements.

Ex M.S. system

position

$$\begin{bmatrix} P(k+1) \\ V(k+1) \end{bmatrix} = \begin{bmatrix} * & * \\ * & * \end{bmatrix} \begin{bmatrix} P(k) \\ V(k) \end{bmatrix}$$

velocity

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} P(k) \\ V(k) \end{bmatrix}$$

Position measurements ✓

unknown velocity

Key question: can we determine initial condition from limited ("potentially" noisy) measurements.

$$y(0) = C\,x(0)$$

$$y(1) = C\,x(1) = CA\,x(0)$$

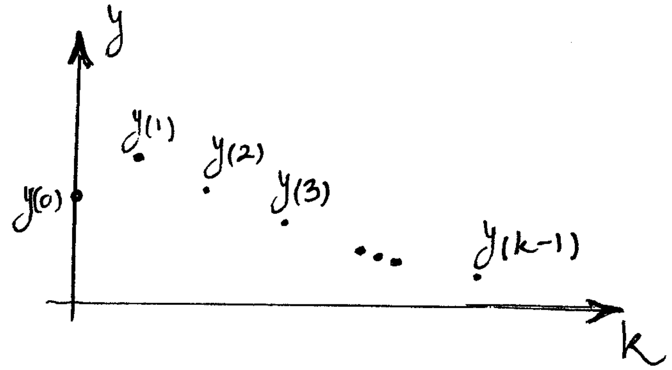
$$\vdots$$

$$y(k-1) = C\,x(k-1) = \cdots = CA^{k-1}x(0)$$

$$\underbrace{\begin{bmatrix} y(0) \\ \vdots \\ y(k-1) \end{bmatrix}}_{\substack{\text{measured} \\ \text{(known)}}} = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{bmatrix}}_{\substack{\text{model} \\ \text{("known")}}} \cdot x(0)$$

unknown

$$\underline{y}_k = O_k\, x(0)$$



$Null(O_k)$: determines initial conditions that cannot be distinguished from zero initial conditions.

* System $x(k+1) = Ax(k)$ is observable
$y(k) = Cx(k)$

$\Updownarrow$ (equivalent to)

$\text{rank}(O_n) = n$

n : # of states

$(Null(O_n) = \{0\})$

$$O_n = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \qquad \text{rank}(O_n) = \text{rank}(O_n^T)$$

$$O_n^T = [\underbrace{C^T}_{\bar{B}} \mid \underbrace{A^T}_{\bar{A}}\underbrace{C^T}_{\bar{B}} \mid \cdots \mid (A^T)^{n-1}C^T]$$

$$= [\bar{B} \mid \bar{A}\bar{B} \mid \cdots \mid (\bar{A})^{n-1}\bar{B}]$$ reachability matrix of system:

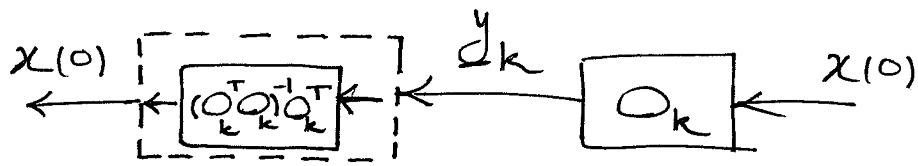
$Z(k+1) = A^T Z(k) + C^T u(k)$

So observability of the system

$$x(k+1) = A\,x(k)$$
$$y(k) = C\,x(k)$$

is equivalent to the reachability of the system

$$z(k+1) = A^T z(k) + C^T u(k)$$



$$y(k) = C\,x(k) + \underbrace{w(k)}_{\text{measurement noise}}$$

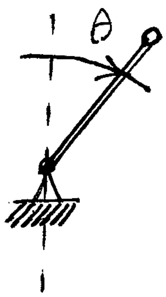
$$\underline{W}_k = \begin{bmatrix} w(0) \\ \vdots \\ w(k-1) \end{bmatrix}$$

$$\underline{y}_k = O_k\, x(0) + \underline{W}_k$$

$$\begin{bmatrix} \underline{y}_k \end{bmatrix} = \underbrace{\begin{bmatrix} O_k \end{bmatrix}}_{\text{skinny matrix}} \begin{bmatrix} x(0) \end{bmatrix} + \begin{bmatrix} \underline{W}_k \end{bmatrix}$$

$$(O_k^T O_k) = \sum_{i=0}^{k-1} (A^i)^T C^T C A^i$$

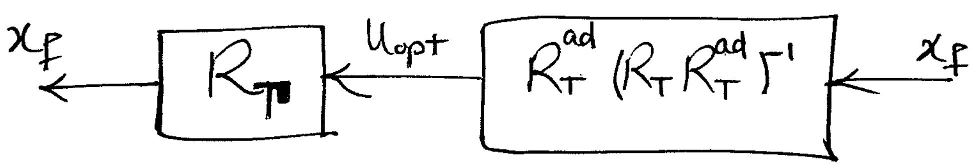
State-feedback design :



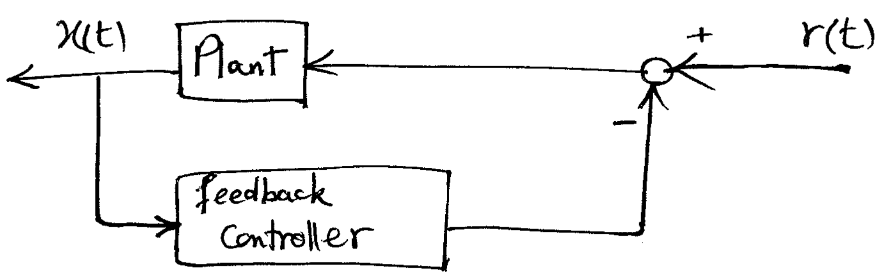
$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

Want to measure $x_1(t)$ and $x_2(t)$ and to use these measurements to achieve some objective

$$u(t) = -[k_1\, k_2] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + r(t)$$ → reference signal



open loop control



K_1 , K_2 : feedback gains

(#'s that determine how we should process our measurements)

$\downarrow$

$x_1(t)$ & $x_2(t)$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1-K_1 & -K_2 \end{bmatrix}}_{} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(t)$$

$$f(s) = \det(sI - A_{cl}) = s(s+K_2) - 1 + K_1$$

$$= s^2 + K_2 s + K_1 - 1 = (s-\lambda_1)(s-\lambda_2)$$