Lecture 24

Linear Systems 12/04/12

Last time

Many things ... (Ctrb, obsv, ...)

Today

- pole placement
- ~~observer design (i.e. state estimation)~~

If time permits : observer-based design (output feedback)

Fact: If system $\dot{x} = Ax + Bu$ (1)

is ctrb. then we can design a state-feedback

controller :

$$u(t) = -k\,x(t) \quad (2)$$

control signal at time t

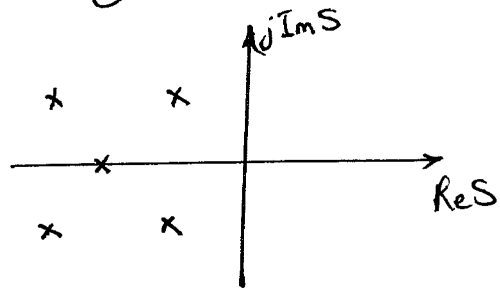
feedback gain matrix [Matrix of feedback gains]

state measurement at time (t)

to place e-values of a closed-loop A-matrix :

$$A_{cl} = A - BK$$

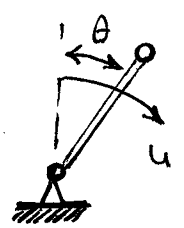
to an arbitrary location in the LHP



(2) → (1) closed-loop system :

$$\dot{x} = Ax + B(-Kx) \Rightarrow \dot{x} = \underbrace{(A - BK)}_{A_{cl}} x$$

Ex.



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$u(t) = -Kx(t) = -[K_1 \ K_2] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$K_1, K_2 \in \mathbb{R}$$

$$A_{cl} = \begin{bmatrix} 0 & 1 \\ 1-K_1 & -K_2 \end{bmatrix}$$

$$f(s) = \det(sI - A) = s^2 + K_2 s + K_1 - 1$$

$$= (s-\lambda_1)(s-\lambda_2) = s^2 - (\lambda_1+\lambda_2)s + \lambda_1\lambda_2$$

(Matlab) use the command: place$(A, B, [\lambda_1, \lambda_2])$
which would K as an output.

* Observer design
(Estimator)

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state eq: $\dot{x} = Ax + Bu$ known input

measured output: $y = Cx$ (*)

Objective: Design a dynamical system [model that we want to simulate]
(observer, estimator)

s.t.

$$\hat{x}(t) \xrightarrow{t \to \infty} x(t)$$

state of estimator (observer)

The problem from simulating this system comes from not knowing the initial conditions of the system $x(0)$

If we knew $x(0)$, we could simulate $(*)$ but we don't (know $x(0)$)

An attempt (at designing an observer) "copy of $(*)$"

$$\left.\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + Bu \quad ; \quad \hat{x}(0) : \text{ under our control} \\ \hat{y}(t) &= C\hat{x}(t) \end{aligned}\right\} \text{(NO)}$$

Naive

If we simulate (NO) will we get $\hat{x}(t) \xrightarrow{t\to\infty} x(t)$?

Form: $\tilde{x}(t) = x(t) - \hat{x}(t)$ and check if $\tilde{x}(t) \xrightarrow{t\to\infty} 0$.

estimation error

$(*) - \text{(NO)}$:

$$\dot{x} - \dot{\hat{x}} = A\cdot(x - \hat{x})$$

$$y - \hat{y} = C(x - \hat{x})$$

$$\dot{\tilde{x}} = A\tilde{x}(t)$$

$$\tilde{y} = C\tilde{x}(t)$$

Q: $\tilde{x}(t) \xrightarrow{t\to\infty} 0$

A: Yes if $\lambda_i(A) \in LHP$ $i = 1, \dots, n$

$$\tilde{x}(0) = x(0) - \hat{x}(0) \neq 0$$

? known

Ex I.P.

$y = x_1$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \text{known} \\ ? \end{bmatrix}$$

$y = x_1$

We will have problems for unstable systems [e.g. inverted pendulum] because of e-values that are not in LHP.

Fix

observer gain matrix

measured output

$$(**) \begin{cases} \dot{\hat{x}}(t) = \underbrace{A\hat{x}(t) + Bu(t)}_{\text{a "copy" of our system}} + \underbrace{L\,(y(t) - \hat{y}(t))}_{\text{an injection term}} \\ \hat{y} = C\hat{x}(t) \end{cases}$$

estimated output

$$(**): \quad \dot{\hat{x}} = A\hat{x} + Bu + LC(x - \hat{x}) \quad \ldots \quad (\mathrm{I})$$

from $(*) - (\mathrm{I})$ we get:

$$\dot{x} - \dot{\hat{x}} = A(x - \hat{x}) + \cancel{Bu} - \cancel{Bu} - LC(x - \hat{x})$$

$$y - \hat{y} = C(x - \hat{x})$$

$$\Rightarrow \qquad \dot{\tilde{x}} = (A - LC)\tilde{x}$$
$$\tilde{y} = C\tilde{x}$$

Conclusion: If $A-LC$ is a stable matrix

$$\tilde{x}(t) \xrightarrow{t\to\infty} 0$$

(our ability to drive $\hat{x}(t)$ to $x(t)$ as $t \to \infty$ depends on whether we could choose L st. $\lambda_i(A-LC) \in LHP$)

Ex. I.P.

$$A - LC = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} -l_1 & 1 \\ 1-l_2 & 0 \end{bmatrix}$$

Vs.

$$A - BK = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} k_1 & k_2 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 1 \\ 1-k_1 & -k_2 \end{bmatrix}$$

Fact: e-values of $A-LC$ $\equiv$ e-values of $\underbrace{A^T}_{\bar{A}} - \underbrace{C^T}_{\bar{B}}\underbrace{L^T}_{\bar{K}} = (A-LC)^T$

e-values don't change by transposing a matrix.

Thus: our ability to move e-values of $A-LC$ to LHP is determined by controlability of pair (A^T, C^T)

$$\Updownarrow$$

observability of "pair" (A, C)

$$\text{rank}\left[\, C^T \,\vdots\, A^T C^T \,\vdots\, \cdots \,\vdots\, (A^{n-1})^T C^T \,\right] = n$$

$$\Updownarrow$$

$$\text{rank}\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n \iff \begin{array}{l} \dot{x} = Ax \\ y = Cx \end{array} \text{ observable!}$$