Lecture 25

12/06/12

Linear Systems

State estimation

(Observer design)

Estimator

$$\left.\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}\right\} \text{ model of your system}$$

P

y → measured output

$$\left.\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L(y-\hat{y}) \\ \hat{y} &= C\hat{x} \end{aligned}\right\} \Rightarrow \quad \dot{\tilde{x}}(t) = (A-LC)\,\tilde{x}(t)$$

If (A,C) observable then L can be selected st. $\tilde{x}(t) \xrightarrow{t\to\infty} 0$

$$\tilde{x}(t) = x(t) - \hat{x}(t)$$

Note! $\tilde{x}(t) = e^{(A-LC)t}\,\tilde{x}(0)$

$\tilde{x}(0) = (x(0) - \hat{x}(0)) \neq 0$

"Bigger L" $\Rightarrow$ faster convergence of $\tilde{x}(0)$ to 0.

Problem: We haven't paid attention to measurement noise.

$$y(t) = Cx(t) + w(t)$$

$$\dot{\hat{x}}(t) \Rightarrow \dot{\hat{x}}(t) = A\hat{x} + Bu + LC(x - \hat{x}) + Lw$$

$$\dot{x} - \dot{\hat{x}} = (A - LC)(x - \hat{x}) - Lw$$

$$\dot{\tilde{x}} = (A - LC)\tilde{x}(t) - Lw(t)$$

$\longrightarrow \cdot \longleftarrow$

trade-off!

We have a trade-off between rate of convergence (of $\hat{x}(t)$ to $x(t)$) and noise amplification

loosely speaking:

big L $\Rightarrow$ faster convergence

small L $\Rightarrow$ small noise amplification

Optimal observer: Kalman filter

this filter essentially minimizes $\lim_{t \to \infty} \underbrace{\mathcal{E}}_{\text{expectation}} \{ \tilde{x}^T(t) \tilde{x}(t) \}$

Observer based design → (Controller design that has an observer as its integral part)

(output feedback design)

Aside

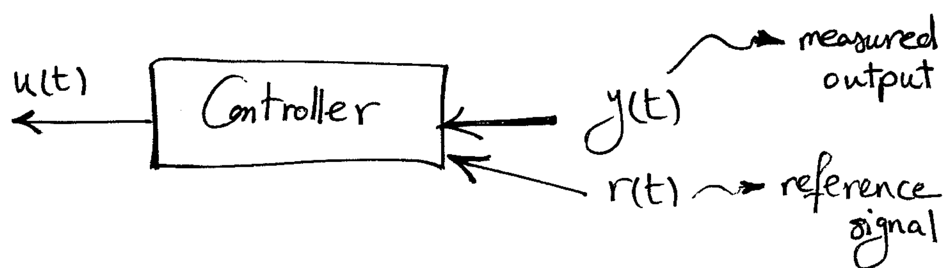
So far: state-feedback design

$$\dot{x} = Ax + Bu$$
$$u = -Kx$$

problem: not all state components are available for measurement.

Propose:

$$\dot{\hat{x}}(t) = A\hat{x} + Bu + L(y - \hat{y})$$
$$u(t) = -K\hat{x}(t) + r(t) \qquad \text{C}$$

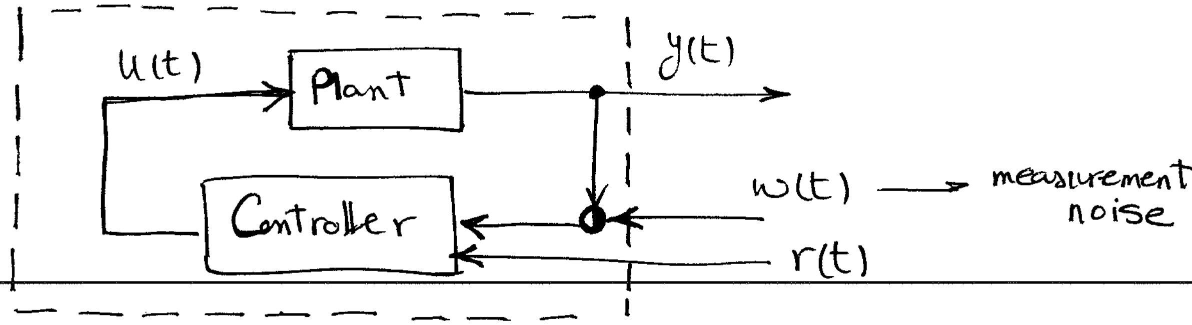


Combine P & C and see what happens.

$$\dot{x} = Ax + B(-K\hat{x} + r)$$
$$\dot{\hat{x}} = A\hat{x} + B(-K\hat{x} + r) + LC(x - \hat{x})$$

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & -BK \\ LC & (A-BK-LC) \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} r(t)$$

$$y(t) = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$



* For what values of L & K do we achieve stability?

In $\begin{bmatrix} x \\ \hat{x} \end{bmatrix}$ - coordinates

Note! Difficult to (see how to) choose L & K s.t.

$$A_{cl} = \left[\begin{array}{c|c} A & -BK \\ \hline LC & A-BK-LC \end{array}\right] \text{ is stable.}$$

$$(\lambda_i(A_{cl}) \in LHP)$$

We'll look at :

$$\begin{bmatrix} x(t) \\ \tilde{x}(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ x(t) - \hat{x}(t) \end{bmatrix} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}$$

$$\dot{x} = Ax - BK(x - \tilde{x}) + Br$$

$$\dot{x} - \dot{\hat{x}} = (A - LC)x - (A - LC)\hat{x} = (A - LC)\tilde{x}(t)$$

$$\Rightarrow \quad \begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} r$$

A, B & C are known
K & L have to be chosen to guarantee stability

Upper block triangular form of $\bar{A}_{cl}$ imply that e-values of $\bar{A}_{cl} = \{\text{e-values of } A - BK\} \cup \{\text{e-values of } A - LC\}$!!!

Summary! If (A,B) is controllable & (A,C) is observable we can choose K & L for $(A-BK)$ and $(A-LC)$ to be stable.

Big conclussion :

If (A,B) is stabalizable and (A,C) is detectable

$$\Downarrow$$

we can design K and L to provide the stability of $(A-BK)$ and $(A-LC)$ and we can provide stability of the closed-loop system (Plant $\oplus$ observer-based controller)