linear plants or system — quadratic performance index — controllers

Linear Quadratic Regulator (LQR)

- **Minimize quadratic objective subject to linear dynamic constraint**

kinetic+potential energy — control effort

$$\text{minimize} \quad J(x,u) = \frac{1}{2}\int_0^T \left(\langle x(\tau), Q\,x(\tau)\rangle + \langle u(\tau), R\,u(\tau)\rangle \right) \mathrm{d}\tau + \frac{1}{2}\langle x(T), Q_T\,x(T)\rangle$$

$$\text{subject to} \quad A\,x(t) + B\,u(t) - \dot{x}(t) = 0$$

$$x(0) = x_0, \quad t \in [0, T]$$

finite time horizon

★ **state and control weights**

$$Q = Q^* \geq 0, \quad Q_T = Q_T^* \geq 0, \quad R = R^* > 0$$

$$\langle x(\tau), Q\,x(\tau)\rangle = x^*(\tau)\,Q\,x(\tau)$$

$$\langle u(\tau), R\,u(\tau)\rangle = u^*(\tau)\,R\,u(\tau)$$

standard quadratic form similar to previous discussions

$$\text{minimize} \quad J(x,u) = \frac{1}{2}\int_0^T \left(\langle x(\tau), Q\,x(\tau)\rangle + \langle u(\tau), R\,u(\tau)\rangle \right) \mathrm{d}\tau \;+\; \frac{1}{2}\langle x(T), Q_T\,x(T)\rangle$$

$$\text{subject to} \quad A\,x(t) + B\,u(t) - \dot{x}(t) = 0$$

$$x(0) = x_0, \quad t \in [0,\,T]$$

- **Features**
  - ⋆ **optimization variable is a function** (not a vector)

    control signal ← $u\colon\; [0,\,T] \longrightarrow \mathbb{R}^m$

    Paying more attention to x_f rather than keeping input effort small.

    $Q \gg R$

  - ⋆ **state and control weights**

    how important final state is! ← $Q,\ Q_T$ symmetric, positive semi-definite

    how important amount of control input is! ← R symmetric, positive definite

  - ⋆ **infinite number of constraints**

- **Introduce Lagrangian**

get rid of time dependence by integrating over time

lagrange multiplier (the price we are paying for violating constraints!)

$$\mathcal{L}(x,u,\lambda) \;=\; J(x,u) \;+\; \int_0^T \langle \lambda(\tau), A\,x(\tau) \,+\, B\,u(\tau) \,-\, \dot{x}(\tau) \rangle \, \mathrm{d}\tau$$

$\star$ **form variations wrt** x, u, λ

$$\mathcal{L}(x, u + \tilde{u}, \lambda) \,-\, \mathcal{L}(x,u,\lambda) \;=\; \int_0^T \langle R\,u(\tau) \,+\, B^*\lambda(\tau), \tilde{u}(\tau) \rangle \, \mathrm{d}\tau \;=\; 0$$

Variation with respect to u

$$\Downarrow$$

$$u(t) \;=\; -R^{-1} B^* \lambda(t), \quad t \in [0,\, T]$$

necessary conditions for optimality:

$$\boxed{\begin{array}{llllll}
\text{wrt } \lambda & \Rightarrow & \dot{x}(t) = A\,x(t) + B\,u(t), & & x(0) = x_0 \\
\text{wrt } x & \Rightarrow & \dot{\lambda}(t) = -Q\,x(t) - A^*\lambda(t), & & \lambda(T) = Q_T\,x(T) \\
\text{wrt } u & \Rightarrow & u(t) = -R^{-1}B^*\lambda(t), & & t \in [0,\, T]
\end{array}}$$

Solution to finite horizon LQR

two-point boundary value problem: (see notes)

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\lambda}(t) \end{bmatrix} = \begin{bmatrix} A & -B\,R^{-1}B^* \\ -Q & -A^* \end{bmatrix} \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix}$$

$$\begin{bmatrix} x_0 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}}_{N_1} \begin{bmatrix} x(0) \\ \lambda(0) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ Q_T & -I \end{bmatrix}}_{N_2} \underbrace{\begin{bmatrix} x(T) \\ \lambda(T) \end{bmatrix}}_{z(T)} \longrightarrow (*)$$

$$u(t) \;=\; -R^{-1}B^*\lambda(t)$$

- **Differential Riccati Equation**

matrix that relates state with λ.
(the same size of A matrix)

can show: $\lambda(t) \;=\; X(t)\,x(t)$

$$-\dot{X}(t) \;=\; A^*X(t) + X(t)\,A + Q - X(t)\,B\,R^{-1}B^*X(t)$$
$$X(T) \;=\; Q_T \qquad \text{(from(*))}$$

$\star$ **optimal controller: determined by state-feedback**

$$u(t) \;=\; -K(t)\,x(t)$$

$$K(t) \;=\; R^{-1}B^*X(t) \longrightarrow \text{can be computed offline}$$

and apply u(t) as we propogate in time by measuring state $x(t)$

Infinite horizon LQR

$$\text{minimize} \quad J = \frac{1}{2} \int_0^\infty \left(\langle x(\tau), Q\, x(\tau) \rangle + \langle u(\tau), R\, u(\tau) \rangle \right) \mathrm{d}\tau$$

$$\text{subject to} \quad \dot{x}(t) = A\, x(t) + B\, u(t)$$

- **Optimal controller:** $\left\{ \begin{aligned} u(t) &= -K\, x(t) \\ K &= R^{-1} B^* X \end{aligned} \right.$

no longer a function of time

$\star$ $X = X^*$ – **non-negative solution to Algebraic Riccati Equation (ARE)**

$$A^* X + X A + Q - X B R^{-1} B^* X = 0$$

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all unstable modes are controllable

$$\left. \begin{array}{ll} (A, B) & \text{stabilizable} \\ (A, Q) & \text{detectable} \end{array} \right\} \Rightarrow \text{stability of } \dot{x}(t) = (A - B K)\, x(t)$$

all unstable modes are observable in index of our cost functional.

$\rightarrow y(t) = Q^{1/2} x(t)$

$\int_0^\infty x^T(\tau) Q x(\tau) d\tau = \int_0^\infty x^T(\tau) Q^{1/2} Q^{1/2} x(\tau) d\tau = \int_0^\infty y^T(\tau) y(\tau) d\tau$

$A^TP + PA + Q - PBR^{-1}B^TP = 0$

$R = r > 0$

$Q = q > 0$

$A = a$

$B = 1$

Scalar example

$$\dot{x} = a\,x + u$$

$$J = \frac{1}{2}\int_0^\infty (q\,x^2(\tau) + r\,u^2(\tau))\,\mathrm{d}\tau$$

$\frac{1}{r}P^2 - 2aP - q = 0$

$P_{1,2} = \frac{2a \pm \sqrt{4a^2 + 4/r\, q}}{2/r} = r\{a \pm \sqrt{a^2 + q/r}\}$

select + : $K_{opt} = \frac{1}{r}b'P = a + \sqrt{a^2 + \frac{q}{r}} \Rightarrow \boxed{a_{cl}} = a - bK = \boxed{-\sqrt{a^2 + \frac{q}{r}}}$

- **Optimal controller**

$$k_{\mathrm{lqr}} = a + \sqrt{a^2 + \frac{q}{r}} \;\Rightarrow\; x(t) = \exp\left(-\sqrt{a^2 + \frac{q}{r}}\,t\right) x(0)$$

tradeoff:

	large q/r	small q/r
convergence rate	fast ✓	slow
control effort	large	low ✓

Matlab: lqr $\rightarrow \infty$ horizon LQR

are : care , dare

care $\downarrow$ Continuous time

dare $\rightarrow$ discrete time

State-feedback H_2 controller

minimize $\quad \lim_{t \to \infty} \mathcal{E}\big(\langle x(t), Q\, x(t) \rangle \, + \, \langle u(t), R\, u(t) \rangle \big)$

subject to $\quad \dot{x}(t) \; = \; A\, x(t) \, + \, B_d\, d(t) \, + \, B_u\, u(t)$

$$\mathcal{E}\left(d(t_1)\, d^*(t_2)\right) \; = \; W_d\, \delta(t_1 \, - \, t_2)$$

expectation operator

(white in time)
no correlation between two different time instances.

- **Minimum variance controller**

state-feedback controller:

$$\boxed{\begin{aligned} u(t) \; &= \; -K\, x(t) \\ K \; &= \; R^{-1} B_u^*\, X \\ 0 \; &= \; A^*\, X \, + \, X\, A \, + \, Q \, - \, X\, B_u\, R^{-1} B_u^*\, X \end{aligned}}$$

State estimation

state equation: $\dot{x}(t) = A\,x(t) + B_d\,d(t) + B_u\,u(t)$

measured output: $y(t) = C\,x(t) + n(t)$

$d(t)$ – process disturbance; $n(t)$ – measurement noise

- **Estimator (observer)**

⋆ **copy of the system + linear injection term**

$$\dot{\hat{x}}(t) = A\,\hat{x}(t) + 0 \cdot d(t) + B_u\,u(t) + L\,(y(t) - \hat{y}(t))$$
$$\hat{y}(t) = C\,\hat{x}(t) + 0 \cdot n(t)$$

we don't know d(t) and n(t) so we cannot copy it.

⋆ **estimation error:** $\tilde{x}(t) = x(t) - \hat{x}(t)$

$$\dot{\tilde{x}}(t) = (A - L\,C)\,\tilde{x}(t) + \begin{bmatrix} B_d & -L \end{bmatrix} \begin{bmatrix} d(t) \\ n(t) \end{bmatrix}$$
$$\tilde{y}(t) = C\,\tilde{x}(t) + n(t)$$

(A, C): detectable $\Rightarrow$ can design L to provide stability of the error dynamics

Kalman filter

(optimal observer in presence of white in time disturbance & noise)

$$\dot{x}(t) = A\,x(t) + B_d\,d(t) + B_u\,u(t)$$
$$y(t) = C\,x(t) + n(t)$$

$$\mathcal{E}\,(d(t_1)\,d^*(t_2)) = W_d\,\delta(t_1 - t_2); \quad \mathcal{E}\,(n(t_1)\,n^*(t_2)) = W_n\,\delta(t_1 - t_2)$$

- **Kalman filter: optimal estimator**
  - ⋆ **minimizes steady-state variance of** $\tilde{x}(t) = x(t) - \hat{x}(t)$

Kalman gain:

$$\boxed{\begin{aligned} L &= Y\,C^*\,W_n^{-1} \\ 0 &= A\,Y + Y\,A^* + B_d\,W_d\,B_d^* - Y\,C^*\,W_n^{-1}\,C\,Y \end{aligned}}$$

Output-feedback H_2 controller

$$\text{minimize} \quad \lim_{t \to \infty} \mathcal{E}\left(\langle x(t), Q\, x(t) \rangle + \langle u(t), R\, u(t) \rangle \right)$$

$$\begin{aligned} \text{subject to} \quad \dot{x}(t) &= A\, x(t) + B_d\, d(t) + B_u\, u(t) \\ y(t) &= C\, x(t) + n(t) \end{aligned}$$

$$\mathcal{E}\left(d(t_1)\, d^*(t_2)\right) = W_d\, \delta(t_1 - t_2); \quad \mathcal{E}\left(n(t_1)\, n^*(t_2)\right) = W_n\, \delta(t_1 - t_2)$$

- **Minimum variance controller**

observer-based controller:

inputs

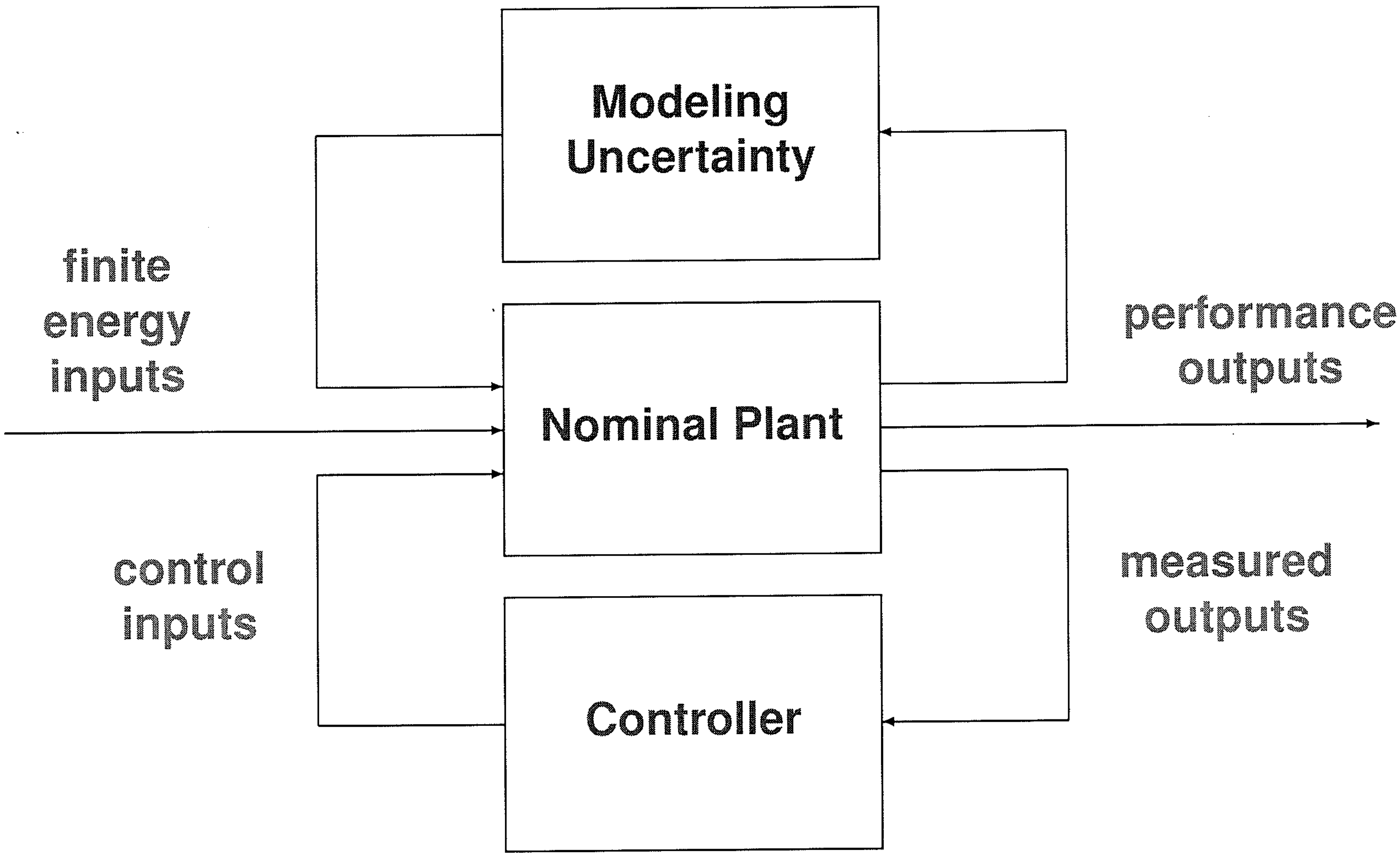
measured outputs

$$\boxed{\begin{aligned} \dot{\hat{x}}(t) &= (A - L\, C)\, \hat{x}(t) + B_u\, u(t) + L\, y(t) \\ u(t) &= -K\, \hat{x}(t) \end{aligned}}$$

$\star$ **feedback and observer gains:** $\begin{cases} K & \text{LQR gain} \\ L & \text{Kalman gain} \end{cases}$ (from sol'n of ARE)

H_∞ controller

- BLENDS CLASSICAL WITH OPTIMAL CONTROL



Take Gary's Robust Control course (EE/AEM 5235) next semester

$$J(x,u) = \frac{1}{2}\int_0^T \left(x^T(\tau) Q x(\tau) + u^T(\tau) R u(\tau)\right) d\tau + \frac{1}{2} x^T(T) Q_T x(T)$$

$$0 = Ax + Bu - \dot{x}$$

$$\mathcal{L}(x,u,\lambda) = J(x,u) + \int_0^T \lambda^T(\tau) \left(A x(\tau) + B u(\tau) - \dot{x}(\tau)\right) d\tau$$

Variation w.r.t. λ:

$$\underbrace{\mathcal{L}(x,u,\lambda+\tilde{\lambda})}_{\text{fixed}} - \mathcal{L}(x,u,\lambda) = \int_0^T \tilde{\lambda}^T(\tau) \underbrace{\left(A x(\tau) + B u(\tau) - \dot{x}(\tau)\right)}_{\equiv 0} d\tau = 0$$

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assume: $z = \begin{bmatrix} x \\ \lambda \end{bmatrix}$

$$\dot{z}(t) = \bar{A} \cdot z(t)$$

$$\underline{b} = N_1 z(0) + N_2 z(T)$$

$N_1 = \begin{bmatrix} & \end{bmatrix}$ slides

$$z(t) = e^{\bar{A}t} z(0) = e^{\bar{A}t} \begin{bmatrix} x(0) \\ \lambda(0) \end{bmatrix}$$

unknown

$$\underline{b} = N_1 z(0) + N_2 e^{\bar{A}T} z(0)$$

$$\underline{b} = (N_1 + N_2 e^{\bar{A}T}) z(0)$$

$$z(0) = (N_1 + N_2 e^{\bar{A}T})^{-1} \underline{\mathbf{b}}$$

$$\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = e^{\bar{A}t} (N_1 + N_2 e^{\bar{A}T})^{-1} \begin{bmatrix} x_0 \\ 0 \end{bmatrix}$$

$$\lambda(t) = f(x_0)$$

$$y(t) = Q^{\frac{1}{2}} x(t)$$

$$\int_0^\infty x^T(\tau) Q x(\tau) d\tau = \int_0^\infty x^T(\tau) Q^{\frac{1}{2}} Q^{\frac{1}{2}} x(\tau) d\tau$$

$$= \int_0^\infty y^T(\tau) y(\tau) d\tau$$