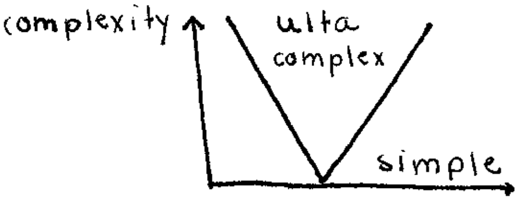
9/5 lecture 2

last time: course mechanics + what/why

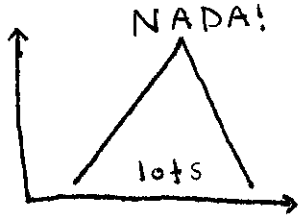
today: basic system properties
1. linearity
2. time invariance
3. causality
4. memory (static vs dynamic)

intro to state-space model

Class of all models:



what can be said? →



1. Linearity (Properties)

a.) homogeniety:

new input $\bar{u} = \alpha \cdot u$ new output $\bar{y} = \alpha \bar{y}$

in other words, if $y = Su$ then $\bar{y} = S\bar{u} = S \cdot \alpha \cdot u = \alpha \cdot S u = \alpha y$

b.) additivity:

$\bar{y} = S[u_1 + u_2] = Su_1 + Su_2 = y_1 + y_2$

⇒ system S is linear if a.) and b.) hold

⇒ $\bar{y} = S[\alpha \cdot u_1 + \beta u_2] = \alpha S u_1 + \beta S u_2 = \alpha \cdot y_1 + \beta \cdot y_2$

last class: saturation is nonlinear

$\left.\begin{array}{l}\text{mass-spring}\\ \text{heat eq}\end{array}\right\}$ linear system

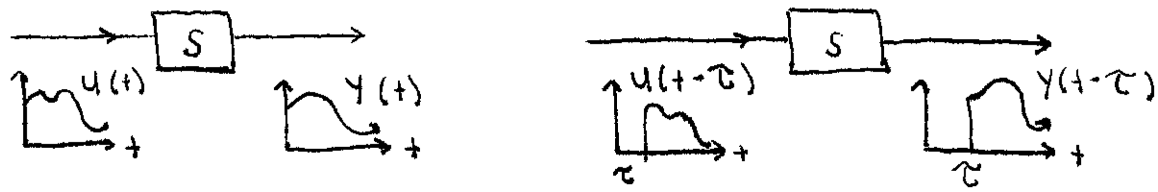
note: $y(t) = \int_{-\infty}^{+\infty} H(t,\tau) \cdot u(\tau)\, d\tau$

+

$y(t) = \sum_{-\infty}^{+\infty} H(t,\tau) \cdot u(\tau)$ are linear!

$\int$ + Σ are linear operators

2. Time Invariance



shift operator: $[\mathcal{T}_\tau u](t) = u(t-\tau)$
$y(t) = [Su](t)$

new input: $\bar{u}(t) = [\mathcal{T}_\tau u](t)$
$\bar{y}(t) = [S\bar{u}](t) = [S\mathcal{T}_\tau u](t)$
$= \mathcal{T}_\tau [Su](t)$
$= [\mathcal{T}_\tau \cdot y](t) = y(t-\tau)$

for all τ

if this holds, S commutes with $\mathcal{T}_\tau$.
$S\mathcal{T}_\tau = \mathcal{T}_\tau S$

note: inverse of $\mathcal{T}_\tau$ is $\mathcal{T}_{-\tau}$ + $\mathcal{T}_{-\tau} S \mathcal{T}_\tau = S$

ex// R(t), i(t)

$$\underset{y(t)}{V(t)} = \underbrace{R(t)}_{S} \cdot \underset{u(t)}{i(t)}$$

$\bar{u}(t) = u(t-T)$
$\bar{y}(t) = R(t)\cdot u(t-T)$
Q: is $\bar{y}(t) = y(t-T)$? $\forall T \in ¢$
$y(t-T) = R(t-T)\cdot u(t-T)$
$R(t)\cdot u(t-T) \overset{?}{=} R(t-T)\cdot u(t-T)$

→ no! doesn't hold unless $R(t)$ = constant

under what conditions is $y(t) = \int_{-\infty}^{+\infty} H(t,\tau)\, u(\tau)\, d\tau$ (CT) and
DT → $y(t) = \sum_{-\infty}^{+\infty} H(t,T)\, u(T)$ invariant wrt time?
⇒ note: input + output mappings (CT) + (DT) are time-invariant if $H(t,\tau) = H(t,-\tau)$

→ $y(t) = \sum_{-\infty}^{\infty} H(t-\tau)\cdot u(\tau)$; $t \in \mathbb{Z}$ (integers)

$$\begin{bmatrix} \vdots \\ y_{-1} \\ y_0 \\ y_1 \\ \vdots \end{bmatrix} = \begin{bmatrix} \ddots & & & \\ & H(-1,-1) & H(-1,0) & H(-1,1) \\ & H(0,-1) & H(0,0) & H(0,1) \\ & H(1,-1) & H(1,0) & H(1,1) \\ & & & & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ u_{-1} \\ u_0 \\ u_1 \\ \vdots \end{bmatrix}$$

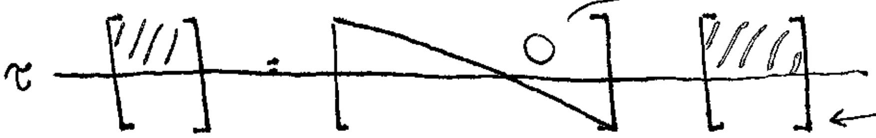
* note: H is constant along diagonal!

for time invariant, $H(t,\tau) = H(t,-\tau)$

3. Causality
↳ output at a certain time only depends on inputs up until that time

ex// $y(t) = \int_{-\infty}^{\infty} H(t-\tau)\, u(\tau)\, d\tau$

otherwise we have future stuff here ;



$H(t,\tau) = 0$ if $t < \tau$

for time invariant: $H(t, -\tau) = 0,\ t < \tau$

$H(\xi) = 0,\ \xi < 0,\ t - \tau < 0$

for time invariant systems:

~~inves~~ impulse response $\equiv 0$ for negative time

4. Memory (static vs. dynamic)

def of static system: output depends only on current value of input

↳ ex// lever, resistance

ex// mass-spring: $m\ddot{y} + ky = u$

↳ $y(\tau) = f(u[0,\tau];\ y(0),\ \dot{y}(0))$

ex// $y(t) = \int_{-\infty}^{\infty} H(t-\tau)\, u(\tau)\, d\tau$ → is it static?

↳ dynamic unless $H(t-\tau) = \delta(t-\tau)$

State Space Models:

$dx/dt = f(x, u, t)$

$y = g(x, u, t)$

where

t: time

$x(t) \in \mathbb{R}^n$: state vector

$u(t) \in \mathbb{R}^m$: input vector

$y(t) \in \mathbb{R}^p$: output

f, g in general, nonlinear function of its arguments.

(1). state equation [1st order in time differential eq]

(2). output - " - [static in time eq]