# Lecture 16 10/29

Last time: Lyapanov Function

↳ scalar functions of State $x$

1.) $V(x)$ is globally positive definite

$V(0) = 0$ & $V(x) > 0 \quad \forall\ x \neq 0$

2.) $V(x)$ is radially unbounded

$\lim_{\|x\| \to \infty} V(x) \longrightarrow \infty$

3.) $dV(x)/d(t) = dV/dx \cdot \frac{dx}{dt}$ is negative definite

$\Rightarrow$ then $\bar{x} = 0$ of $\dot{x} = f(x)$ is globally assymptotically stable

Today: Lyapanov Functions for LTI system

$V(x) = x^T P x \quad ; \quad P = P^T > 0$ (positive definite)

translation $\rightarrow \lambda_i(P) > 0,\ i = 1, \dots, n$ or $x^T P x > 0$ for all $x \neq 0$)

"Big THM":

a system (LTI) is stable (all e-values are in LHP
$\dot{x} = Ax$ $\qquad$ $Re(\lambda_i) < 0$)

if and only if for all $Q = Q^T > 0$, there exists
$P = P^T > 0$ such that $A^T P + PA = -Q$. (ALE)

~~more~~ over, the Lyapanov function is given by

$V(x) = x^T P x$

where

$P = \int_0^\infty e^{A^T t} Q e^{At}\, dt$ is the unique solution to ALE

Proof: "⇑" Assume that for every $Q = Q^T > 0$ there is $P = P^T > 0$
such that $A^T P + PA = -Q$

Propose Lyapanov function candidate:

$V(x) = x^T P x$

$\Rightarrow V(x)$ satisfies 1.) + 2.)

(globally positive def. + radially unbounded)

thus compute

$$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x}$$
$$= x^T (A^T P + PA) x$$
$$= x^T (Q) x$$

It follows that $x^T Q x < 0 \quad \forall\ x \neq 0$.

proof: continued

thus 3.) holds (i.e $\dot{V}(x)$ is globally negative def)

$\Rightarrow \bar{x} = 0$ is GAS

$\Rightarrow \dot{x} = Ax$ is stable

"$\Downarrow$" Assume that $\dot{x} = Ax$ is stable.

then we will show that

* $P = \int_0^\infty e^{A^T t} Q e^{At} \, dt$ w/ $Q = Q^T > 0$

has the following properties.

1. $P = P^T$
2. $P$ is positive def
3. $P$ solves ALE ( $P$ is $\underline{a}$ solution to ALE)
4. $P$ is unique solution to ALE.

1.) $P^T = \left( \int_0^\infty e^{A^T t} Q e^{At} dt \right)^T$

$= \int_0^\infty e^{A^T t} Q^T e^{At} dt$ $\quad$ $*(ABC)^T = C^T B^T A^T$

$Q = Q^T$ thus

$\int_0^\infty e^{A^T t} Q e^{At} dt = P$

thus $P^T = P$.

2.) Compute

$x^T P x = \int_0^\infty x^T e^{A^T t} Q^{\frac{1}{2}} Q^{\frac{1}{2}} e^{At} x \, dt$

call $Q^{\frac{1}{2}} e^{At} x = z(t)$

and $x^T e^{A^T t} Q^{\frac{1}{2}} = z^T(t)$

(used the fact that $Q = Q^T > 0$ has a "square root"
Matlab: sqrtm(Q))

thus

$x^T P x = \int_0^\infty \underbrace{z^T(t) \cdot z(t)}_{\|z(t)\|_2^2} dt \geq 0$

alternitively:
$x^T P x = \int_0^\infty y^T(t) \, Q \, y(t) \, dt$
$\geq 0$
$\Rightarrow y(t) = e^{At} \cdot x$

(Aside $\rightarrow$ can $x^T P x = 0$ for $\bar{x} \neq 0$ ? NO!!!
$e^{At}$ is invertible $\Rightarrow x = (e^{At})^{-1} \cdot y(t)$

3.) Plug * into ALE & see if it holds.

$A^T P + PA = \int_0^\infty (A^T e^{A^T t} Q \cdot e^{At} + e^{A^T t} Q e^{At} \cdot A) \, dt$

(recall definition of $e^{At}$ : $de^{At}/dt = A \cdot e^{At} = e^{At} A$ )

$= \int_0^\infty \left( de^{A^T t}/dt \cdot Q e^{At} + e^{A^T t} Q \, de^{At}/dt \right) dt$

$= \int_0^\infty \frac{d}{dt} ( e^{A^T t} Q e^{At} ) \, dt = (e^{A^T t} Q e^{At}) \Big|_0^\infty = \lim_{t \to \infty} \cancel{e^{A^T t}} \xrightarrow{A\text{-stable}} Q e^{At} - e^{A^T 0} Q e^{A0}$

$= (0 - I) Q I = -Q$

proof : continued

4.) $\left.\begin{array}{l} P_1 = * \\ P_2 = * \end{array}\right| \Rightarrow \begin{array}{l} A^T P_1 + P_1 A = -Q \\ A^T P_2 + P_2 A = -Q \end{array}$

$$\Rightarrow A^T \underbrace{(P_1 - P_2)}_{M} + \underbrace{(P_1 - P_2)}_{M} A = 0 \cdot Q$$

$$M = \int_0^\infty e^{A^T t} \cdot 0 \cdot Q \cdot e^{At}\, dt = 0$$

the end ☺

remember to revisit after midterm

Remember : $A^T P + PA = -Q$

choose <u>any</u> Q, if system stable... etc etc.

ex// $A = \begin{bmatrix} -1 & 0 \\ k & -2 \end{bmatrix}$

propose $P = P^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A^T P + PA = (A^T + A) = \begin{bmatrix} -2 & k \\ k & -4 \end{bmatrix} = -\begin{bmatrix} 2 & -k \\ -k & 4 \end{bmatrix}$

$\Delta_1 (Q) = 2 > 0$

$\Delta_2 (Q) = 8 - k^2 > 0 \Leftrightarrow |k| < 2\sqrt{2}$