Nonlinear Systems

Lecture 07 02/12/13

Last time:

- Hopf bifurcation
- Scaling / Non-dimensionalization

Today:

- Center manifold theory ⟶ (Chapter 8) ← Khalil
- Existence/uniqueness of sol'n

Comment on HW2 / Q3b

$$\ddot{y} + \alpha h'(y)\dot{y} + \beta y = 0$$

$$\left.\begin{aligned} x_1 &= y \\ x_2 &= \dot{y} + \alpha h(y) \end{aligned}\right\} \Rightarrow \begin{aligned} \dot{x}_1 &= \dot{y} = -\alpha h(x_1) + x_2 \\ \dot{x}_2 &= \ddot{y} + \alpha h'(y)\dot{y} = -\beta x_1 \end{aligned}$$

$$V(x) = \frac{1}{2}(\beta x_1^2 + x_2^2)$$

Center manifold theory

$$\dot{x} = f(x) \qquad (1)$$

$x(t) \in \mathbb{R}^n$:

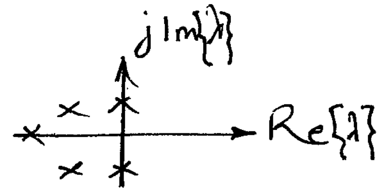
$f(0) = 0 \Rightarrow \bar{x} = 0$ is an e.p.

Assume that linearization around $\bar{x}=0$ has

K e-values on $j\omega$-axis

n-K e-values in the LHP

($Re\,\lambda < 0$)



We'll rewrite (1) as:

$$\dot{x} = Ax + \tilde{f}(x) \qquad \text{where} \qquad \tilde{f} = f(x) - \left.\frac{\partial f}{\partial x}\right|_{\bar{x}=0} \cdot x$$

Taylor series of f around $\bar{x}=0$

$$f(x) = f(0) + \underbrace{\left.\frac{\partial f}{\partial x}\right|_{0}}_{A} \cdot x + \underbrace{\boxed{H.O.T}}_{\tilde{f}}$$

Note!

$$\tilde{f}(x) = f(x) - \left.\frac{\partial f}{\partial x}\right|_{\bar{x}=0} \cdot x \Rightarrow \tilde{f}(0) = 0$$

$$\frac{\partial \tilde{f}}{\partial x} = \frac{\partial f}{\partial x} - \frac{\partial f}{\partial x}\bigg|_{\bar{x}=0} \Rightarrow \frac{\partial \tilde{f}}{\partial x}\bigg|_{\bar{x}=0} = 0$$

$$\boxed{\frac{\partial \tilde{f}(0)}{\partial x} = 0}$$

So we could rewrite (1) as :

$$\dot{x} = Ax + \tilde{f}(x) \qquad (2)$$

where

$$\tilde{f}(0) = 0 \quad ; \quad \frac{\partial \tilde{f}}{\partial x}\bigg|_0 = 0$$

Introduce a change of coordinates :

$$\begin{bmatrix} y \\ z \end{bmatrix} = T \cdot x \quad ; \quad \begin{array}{l} y(t) \in \mathbb{R}^k \\ z(t) \in \mathbb{R}^{n-k} \end{array}$$

to bring (2) into the following form :

$$\dot{y} = A_1 y + g_1(y,z)$$
$$\dot{z} = A_2 z + g_2(y,z)$$

where (a) A_1 contains e-values on $j\omega$-axis and A_2 contains LHP e-values.

and

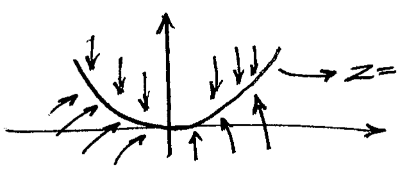
(b) $g_i(0,0) = 0 \quad ; \quad i = 1,2$

$$\left.\frac{\partial g_i}{\partial y}\right|_{(0,0)} = 0 \quad ; \quad \left.\frac{\partial g_i}{\partial z}\right|_{(0,0)} = 0$$

when you start on this surface you stay on it forever

Fact (Thm) : There is an invariant manifold $z = h(y)$ in the neighborhood of the origin that satisfies

$$h(0) = 0 \quad \left.\frac{\partial h}{\partial y}\right|_0 = 0$$



$z = h(y)$, $z - h(y) = 0$

Example : 1D bifurcations in higher dimensions

$$\dot{y} = 0.y + g(y, \alpha, z)$$
$$\dot{\alpha} = 0$$
$$\dot{z} = A_2 z + g_2(y, \alpha, z)$$

$$z = h(y, \alpha)$$

$$\boxed{\begin{aligned} \dot{y} &= 0.y + g_1(y, \alpha, h(y,\alpha)) \\ \dot{\alpha} &= 0 \end{aligned}}$$

Main result:

If the origin of the reduced system,

$$\dot{y} = A_1 y + g_1(y, h(y))$$

is asymptotically stable (unstable) then the origin of

$$\dot{y} = A_1 y + g_1(y, z)$$
$$\dot{z} = A_2 z + g_2(y, z)$$

is asympt. stable (unstable). $\blacksquare$

Characterization of the center manifold
(i.e. how to find $h(y)$)

Introduce a new variable: $\quad w = z - h(y)$

Since $z = h(y)$ is invariant $\Rightarrow$ $\quad w \equiv 0 \Rightarrow \dot{w} \equiv 0$

$$\boxed{\dot{w} =} \; \dot{z} - \frac{\partial h}{\partial y}\dot{y} = \boxed{A_2 h(y) + g_2(y, h(y)) - \frac{\partial h}{\partial y}\left[A_1 y + g_1(y, h(y))\right] = 0 \quad (*)}$$

this eq'n characterizes the center manifold
→ solve for $h(y)$!

Therefore the center manifold can be obtained by solving this eq'n (Non trivial excercise in general!)

Example

Assume $y(t) \in \mathbb{R}$, scalar

and look for approximate solution to $(*)$.

Taylor series of $h(y)$ around the origin,

$$h(y) = \underbrace{h(0)}_{0} + \underbrace{\frac{\partial h}{\partial y}\Big|_0}_{0} \cdot y + \underbrace{h_2}_{\frac{\partial^2 h}{\partial y^2}\big|_0} y^2 + h_3 y^3 + O(y^4)$$

$$\Rightarrow \boxed{h(y) = h_2 y^2 + h_3 y^3 + O(y^4)} \qquad (\mathrm{I})$$

where h_i are constants. Thus, $h(y)$ contains quadratic and higher order terms.

Ex $\quad \dot{y} = \underbrace{0}_{A_1=0} \cdot y + \underbrace{y \cdot z}_{g_1(y,z)}$

$\dot{z} = \underbrace{-}_{A_2=-1} z + \underbrace{a y^2}_{g_2(y,z)} \quad ; \quad a \neq 0$

$$-h(y) + a y^2 - \frac{\partial h}{\partial y}\left(0 \cdot y + y h(y)\right) = 0 \qquad (\text{II})$$

plug (I) into (II) $\Rightarrow$

$$-h_2 y^2 - h_3 y^3 - O(y^4) + a y^2 - \left[2h_2 y + 3h_3 y^2 + O(y^3)\right] \cdot \left[h_2 y^3 + h_3 y^4 + O(y^5)\right] = 0$$

y^2: $\quad \Rightarrow h_2 = a$

$$\Rightarrow h(y) = a y^2 + O(y^3)$$

We then have,

$$\dot{y} = 0 y + y h(y)$$

$$\dot{y} = 0 \cdot y + a y^3 + O(y^4)$$

We can now determine the stability of the system by studying the sign of a in $\dot{y} = a y^3$

$a < 0$ local asymptotic stability

$a > 0$ unstable

Mathematical preliminaries (background) "Chapter 3 Khalil"

Given, $\dot{x} = f(x)$

$x(0) = x_0$

is there a sol'n? If so is it unique?

Continuous dependence on initial conditions/parameters?

↳ All of these questions make sense on both finite $[0, t_f]$ and infinite $[0, \infty)$ time intervals.

Fact: If f is a continuous function (C^0) then there is a sol'n on $[0, t_f]$ (but it may not be unique)

Ex: $\dot{x} = x^{1/3}$