Nonlinear Systems

Lecture 10 02/21/13

Last time:

Sensitivity wrt. parameters
Sensitivity equations

Today:

Lyapunov stability
Lyapunov functions
(i.e. Lyapunov indirect method for checking stability properties)

Lyapunov stability

$\dot{x} = f(x)$ [time-invariant dynamics (for now)]

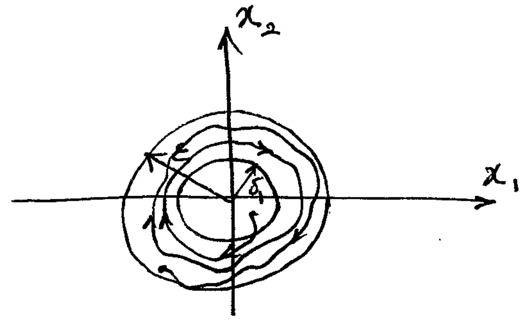
Assume: $f(0) = 0$ (w.l.g.)

Given $\bar{x} = 0$ (e.p. @ the origin), determine qualitative features of the system's response for $x_0 \neq 0$.
(e.g. are we going to stay close to the $\bar{x} = 0$ e.p. if we start close to it $\longrightarrow$ stability)

1) $\bar{x} = 0$ is stable if for every $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$
s.t. $\|x_0\| < \delta_1 \Rightarrow \|x(t)\| < \varepsilon$

If $\|\cdot\| = \|\cdot\|_2$

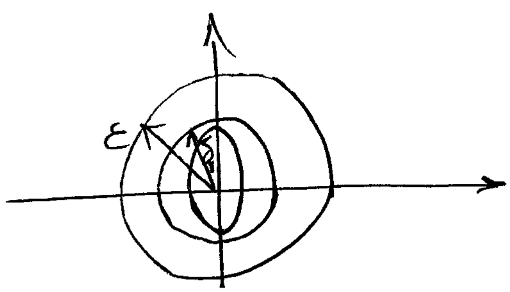
$\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$



Ex. Harmonic oscillator

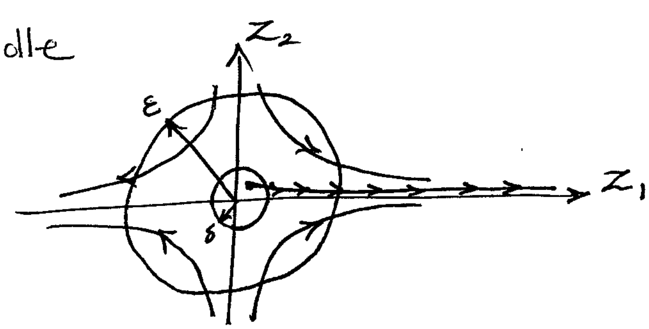
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

MS system
LC circuit
IP linearized around $\bar{x}_{down}$



2) $\bar{x} = 0$ is unstable if it is not stable
(i.e. (1) does not hold)

Ex. Saddle

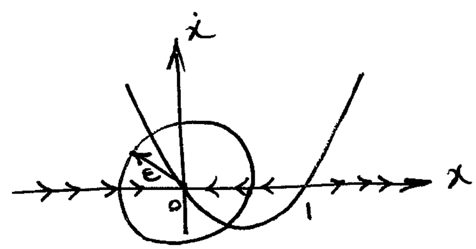


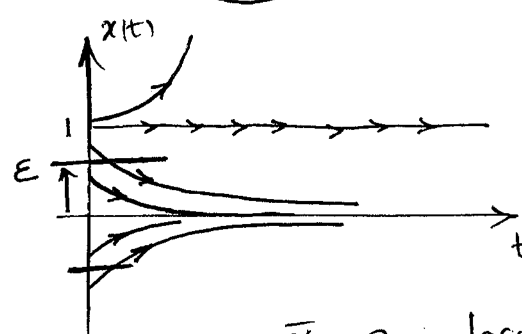
$\bar{x}=0$ is 3) (locally) asymptotically stable if (1) holds and there is $\delta_2>0$ s.t. $\|x_0\|<\delta_2 \Rightarrow \lim_{t\to\infty}\|x(t)\|=0$

4) globally asymptotically stable if (3) holds for $\delta_2=+\infty$ (i.e. for all $x_0\in\mathbb{R}^n$)

Ex. $\dot{x}=x(x-1)$

$\bar{x}=0 \quad \bar{x}=1$





$\bar{x}=0$: locally asymptotically stable

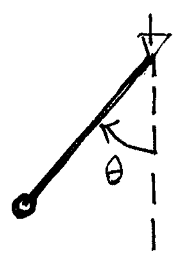
$\bar{x}=1$: unstable

We can use similar arguments for second-order systems (i.e. definition of stability) but this would require knowledge of phase portraits.

Q: Can we check stability properties without finding sol'ns to $\dot{\boldsymbol{x}} = f(x)$?

A: yes, Lyapunov indirect method!

Ex. Pendulum



$x_1 = \theta$

$x_2 = \dot{\theta}$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -a\sin x_1 - b x_2$$

Energy: $\mathbf{E}(t) = a\int_0^{x_1} \sin \xi \, d\xi + \frac{1}{2} x_2^2$

Potential kinetic

$$\dot{E}(t) = a\sin(x_1)\dot{x}_1 + x_2\dot{x}_2 = \cancel{a\sin(x_1)x_2} + x_2(\cancel{-a\sin x_1} - b x_2)$$
$$= -b x_2^2$$

$\frac{dE(t)}{dt} \leq 0 \Rightarrow E(t)$ is a nonincreasing function of time

Note: If $b=0$ (i.e. no damping)

$$\Downarrow$$

$$\frac{dE(t)}{dt} = 0 \Rightarrow E(t) = \text{const.} = E(t_0)$$

(conservative system)

$$= a(1-\cos x_{10}) + \frac{1}{2} x_{20}^2$$

Lyapunov indirect method

Given $V: \mathbb{R}^n \longrightarrow \mathbb{R}$, a continuously differentiable function

$[V(x)$: function of state$]$

s.t.

$$\left.\begin{array}{l} V(0)=0 \\ V(x) > 0 \text{ , for all } x \in \overset{\text{domain}}{D} \setminus \{0\} \end{array}\right\} \text{positive definiteness}$$

then:

1\) If $\underbrace{\dot V(x) \leq 0}_{\text{(negative semidefinite)}}$ $\overset{\text{for all } x \in D\setminus\{0\}}{\Longrightarrow}$ $\bar x = 0$ is stable

2\) If $\dot V(x) < 0$, for all $x \in D \setminus \{0\}$

$$\Downarrow$$

$\bar x = 0$ is (locally) asymptotically stable

3) For global asymptotic stability

we need $D \in \mathbb{R}^n$

$V(x)$: positive definite on $\mathbb{R}^n$

$\dot{V}(x)$: negative definite on $\mathbb{R}^n$

$V(x)$: radially unbounded

$\|x(t)\| \longrightarrow \infty \Rightarrow V(x) \longrightarrow +\infty$

$$\dot{V}(x) = \frac{dV(x)}{dt} = \frac{\partial V(x)}{dx} \frac{dx(t)}{dt} = \nabla f(x) \cdot f(x)$$

where, $\nabla V(x) = \partial V/\partial x = \left[\partial V/\partial x_1 \cdots \partial V/\partial x_n\right]$

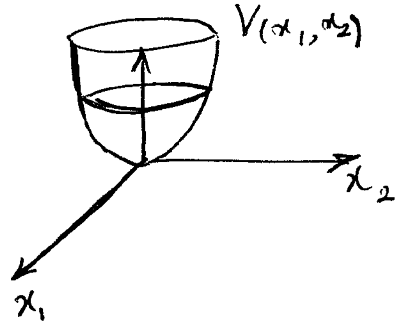
$$f(x) = \begin{bmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{bmatrix}$$

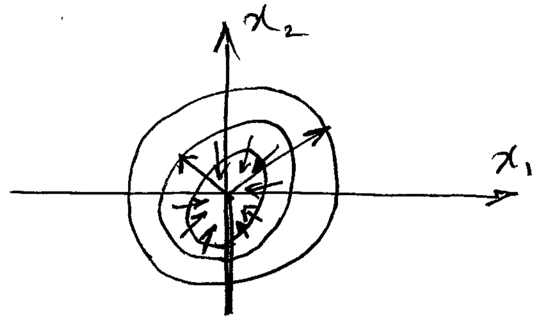
$$\dot{V}(x) = \sum_{i=1}^{n} \partial V/\partial x_i \, f_i(x)$$

Sketch of the proof :

1) $\frac{dV(x)}{dt} \leqslant 0 \Rightarrow \nabla V(x) \cdot f(x) \leqslant 0$

level sets $\{x, V(x) = \tau\}$ are positively invariant


V(x1, x2)
x2
x1


x2
x1