Nonlinear Systems

Lecture 12 02/28/13

Last time

LaSalle's invariance principle
- applies to time-invariant systems
- conditions for asymptotic stability even when $\dot{V}(x) \leq 0$

$\Omega_c = \{ x \ ; \ V(x) \leq c \}$; bounded

$\dot{V}(x) \leq 0$ in Ω_c

$S = \{ x \in \Omega_c \ ; \ \dot{V}(x) = 0 \}$

M : largest invariant set in S $\Rightarrow$ $x(t) \xrightarrow{t \to \infty} M$

$\forall x(0) \in \Omega_c$

Consequence $\Rightarrow$ If $M = \{0\} \Rightarrow$ asymptotic stability of the origin

Ex 1

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -a g(x_1) - b x_2$$

$a > 0$, $b \geq 0$; $g(0) = 0$ $x \cdot g(x) > 0$, for all $x \in (-b, c)$

$$\dot{V}(x) = -0x_1^2 - bx_2^2 = -[x_1 \ x_2]\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq 0$$

$$V(x) = a\int_0^{x_1} g(\xi)\,d\xi + \frac{1}{2}x_2^2$$

$$\Rightarrow \quad S = \{(x_1, x_2),\ x_2 = 0\}$$

$$x_2 \equiv 0 \Rightarrow \dot{x}_2 \equiv 0 \Rightarrow 0 = -ag(x_1) - b.0^2$$

$g(x_1) = 0 \Rightarrow$ from properties of g

$x_1 = 0$

$\Rightarrow M = \{0\} \longrightarrow$ local asymptotic stability

In the linear case:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = ax_1 - bx_2$$

La Salle applies (use argument from Ex1) with:

$$V(x) = ax_1^2 + \tfrac{1}{2}x_2^2$$

In general such Lyapunov functions can be written as:

$V = x^T P x$; In this example $P = \begin{bmatrix} a & 0 \\ 0 & 1/2 \end{bmatrix}$

Note! $\dot{V}$ can be written as

$$\dot{V}(x) = -x^T Q x \quad \text{with} \quad Q = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$$

Lyapunov indirect method for Linear time-invariant systems:

$$\dot{x} = Ax$$

$$V(x) = x^T P x$$

$$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x} = x^T A^T P x + x^T P A x$$

$$= x^T \underbrace{(A^T P + PA)}_{-Q} x$$

$$\boxed{\dot{V}(x) = -x^T Q x}$$

Given $P = P^T > 0$ show that $\lim_{\|x\| \to \infty} x^T P x = +\infty$

$$\lambda_{min}(P) \|x\|^2 \leq x^T P x \leq \lambda_{max}(P) \|x\|^2$$

from positive definiteness of $P = P^T$ (all e-values positive)

$x^T P x > 0, \ \forall x \neq 0$

For linear systems

A: Hurwitz $\left(\mathrm{Re}(\lambda_i(A)) < 0, \ \forall i=1,\dots,n\right)$

$$\Updownarrow$$

For any $Q = Q^T > 0$, there is $P = P^T > 0$ st.

$$A^T P + PA = -Q \quad \blacksquare$$

$$\longleftarrow$$

Note! this P is unique (for any given Q)

$$P = \int_0^\infty e^{A^T t} Q e^{At} \, dt$$

$\downarrow$

observability grammian

(remember w/ A transpose comes first)

$$\underline{\underline{A^T}} P + PA = -Q$$

$$\dot{V}(x) = -x^T Q x$$

$$= -x^T C^T C x = -y^T y$$

La Salle's invariance principle for linear systems:

start with $V(x) = x^T P x$

$\dot{V}(x) = -x^T Q x \leq 0$ on Ω_c (not $<$)

Assume that $\text{rank}(Q) = r < n$. Write $Q = C^T C$ $\quad C \in \mathbb{R}^{r \times n}$ of rank r

$$\dot{V}(x) = -(Cx)^T (Cx) = 0$$

$$\Rightarrow Cx = 0$$

Introduce $y(t) = C \cdot x(t)$

Want, $\quad y = Cx = 0 \Leftrightarrow x = 0$

$$\left. \begin{aligned} \dot{x} &= Ax \\ y &= Cx \end{aligned} \right\} \quad \begin{aligned} &y = 0 \Leftrightarrow x = 0 \\ &x(t) = e^{At} \cdot x_0 \end{aligned}$$

Thus asymptotic stability with $Q = Q^T \geq 0$

$$\Updownarrow$$

$$Q = C^T C$$

Pair (A, C) : observable.

Ex $A = \begin{pmatrix} 0 & 1 \\ -a & -b \end{pmatrix}$ $Q = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$

$$Q = \underbrace{\begin{bmatrix} 0 \\ \sqrt{b} \end{bmatrix}}_{C^T} \underbrace{[0 \ \ \sqrt{b}]}_{C}$$

Observability matrix:

$$\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{b} \\ -a\sqrt{b} & -b\sqrt{b} \end{bmatrix}$$

$$\det\left(\begin{bmatrix} C \\ CA \end{bmatrix} \right) = ab \neq 0$$

$b = 0 \longrightarrow$ conservative system & this analysis would not be needed.

$\Rightarrow$ thus $\boxed{a \neq 0}$ since $a \geq 0 \Rightarrow (a > 0 \Leftrightarrow$ stability$)$

Controllability grammian

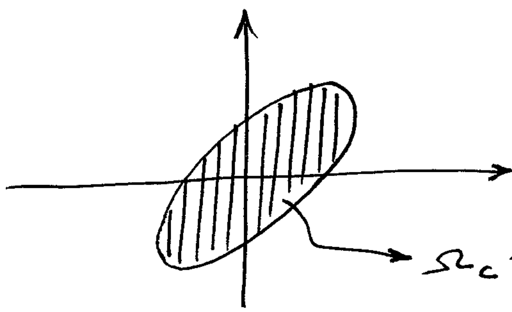
$$AW_c + W_c A^T = -\begin{bmatrix} B \end{bmatrix} [\ B^T\ ]$$

for Observability grammian

$$A^T W_o + W_o A = -\begin{bmatrix} C^T \end{bmatrix}\begin{bmatrix} & C & \end{bmatrix}$$

Region of Attraction:

Set of points s.t. $\{x_0 \ , \ \underbrace{\phi(t, x_0)}_{x(t) \text{ st. } x(0)=x_0} \longrightarrow 0\}$



$\Omega_c = \{x \ ; \ V(x) \leq c\}$

$\dot{V}(x) < 0$ in Ω_c

Q: How to estimate the region of attraction?

A possible approach: use Lyapunov based analysis

(Caveat: Maybe conservative!)

For example: use $V(x) = x^T P x$ where P is <u>a</u> Lyapunov function for corresponding linearization

Use of sum-of-squares (SOS) techniques can reduce conservatism. (Talk to Gary & Pete)

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Lyapunov direct method (Linearization)

LDM

$$\dot{x} = f(x) = Ax + g(x)$$

$$\frac{\|g(x)\|}{\|x\|} \xrightarrow{\|x\| \to 0} 0$$

Choose $V(x) = x^T P x \quad P = P^T > 0$

s.t. $A^T P + PA = -Q \;,\quad Q = Q^T > 0$

$$\dot{V}(x) = -x^T Q x + g^T(x) \cdot P x + x^T P g(x)$$

$$= -x^T Q x + 2x^T P g(x) \leq \underbrace{-x^T Q x}_{\text{negative ☺}} + \underbrace{2\|x\|\|P\|\|g(x)\|}_{\text{positive ☹}}$$

$$\lambda_{min}(Q)\|x\|^2 \leq x^T Q x \leq \lambda_{max}(Q)\|x\|^2$$

$$\begin{aligned} -x^T Q x &\leq -\lambda_{min}(Q)\|x\|^2 \\ &\leq -\lambda_{min}(Q)\|x\|^2 + 2\|x\|\|P\|\|g(x)\| \\ &= -\|x\|^2\left(\lambda_{min}(Q) - 2\|P\|\frac{\|g(x)\|}{\|x\|}\right) \end{aligned}$$

From $\frac{\|g(x)\|}{\|x\|} \xrightarrow{\|x\|\to 0} 0$, for small enough $\|x\|$ the quantity inside the parantheses on the right will be positive.